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INVARIANT TORI FOR THE BILLIARD BALL MAP

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ABSTRACT. For an n-dimensional domain Ω $(n \geq 3)$ with a smooth boundary which is strictly convex in a neighborhood of an elliptic closed geodesic $\mathscr{O}$, the existence of a family of invariant tori for the billiard ball map with a positive measure is proved under the assumptions of nondegeneracy and N-elementarity, $N \geq 5$, of the corresponding to $\mathscr{O}$ Poincaré map. Moreover, the conjugating diffeomorphism constructed is symplectic. An analogous result is obtained in the case $n = 2$. It is shown that the lengths of the periodic geodesics determine uniquely the invariant curves near the boundary and the billiard ball map on them up to a symplectic diffeomorphism.

1. INTRODUCTION

Let Ω be a strictly convex and compact domain in $\mathbb{R}^n$, $n \geq 2$, with a boundary $\partial\Omega$ of class C^∞. In this paper, we investigate the so-called billiard ball map B near the boundary $S^*\partial\Omega$ of the coball bundle $\Sigma = B^*\partial\Omega = \{(x, \xi) \in T^*\partial\Omega;\ |\xi| \leq 1\}$. The billiard ball map is an exact symplectic map in the interior of Σ [6] which is singular on the boundary $\partial\Sigma$. The local behaviour of B near $\partial\Sigma$ was described by Melrose. Using the equivalence theorem for glancing hypersurfaces proved by Melrose [15] one can introduce local symplectic coordinates (x, ξ) in $T^*\partial\Omega$ near any point $\rho \in S^*\partial\Omega$ so that $B^*\partial\Omega = \{\xi_{n-1} \geq 0\}$ and the billiard ball map B assumes the form

$$(x, \xi) \to (x', x_{n-1} - \xi_{n-1}^{1/2}, \xi), \qquad \xi_{n-1} \geq 0,$$

where $x' = (x_1, \dots, x_{n-2})$.

This result is of particular importance for the construction of a local parametrix in Ω for the mixed problem for the wave equation. Our goal in the present paper is to construct global symplectic coordinates near a closed curve lying in $\partial\Sigma$ in which B assumes a "normal form" similar to the one described above.

First we assume that $n \geq 3$. Let $\tilde{\mathscr{O}}$ be a closed bicharacteristic in $S^*\partial\Omega$, i.e. a closed trajectory of the Hamiltonian vector field X_H with Hamiltonian $H(x, \xi) = |\xi|$ for $(x, \xi) \in T^*\partial\Omega$. Denote by $\mathscr{O} \subset \partial\Omega$ the projection of $\tilde{\mathscr{O}}$ on $\partial\Omega$ which turns out to be a closed geodesic on $\partial\Omega$.

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We suppose that $\mathscr{O}$ is elliptic and the corresponding to $\widetilde{\mathscr{O}}$ Poincaré map P is nondegenerate and N-elementary, $N \geq 5$. Our aim is to prove the existence of a family of invariant with respect to B submanifolds $\Lambda_I \subset \Sigma$ near $\widetilde{\mathscr{O}}$, diffeomorphic to the $(n-1)$-dimensional torus $\mathbb{T}^{n-1} = \mathbb{R}^{n-1}/(2\pi\mathbb{Z})^{n-1}$, which are enumerated by I belonging to a Cantor set E with a positive Lebesgue measure in $\mathbb{R}^{n-1}$. We shall construct a smooth function K in $\mathbb{R}^{n-1}$, $K(0) = 0$, $\operatorname{grad} K(0) \neq 0$ and an exact symplectic transformation U from a neighbourhood of $\widetilde{\mathscr{O}}$ into $T^*(\mathbb{T}^{n-1})$ mapping Λ_I into $\mathbb{T}^{n-1} \times \{I\}$ for $I \in E$ such that B assumes the following "normal form"

$$B(\varphi, I) = (\varphi - \operatorname{grad}(\tfrac{2}{3}(K(I))^{3/2}), I)$$

for any $\varphi \in \mathbb{T}^{n-1}$ and $I \in E$.

The existence of a family of invariant tori for the billiard ball map in the case $n = 3$ was announced by Svanidze in [20]. In contrast to [20] we construct the conjugating diffeomorphism U symplectic and describe precisely the singularity of B near $\partial\Sigma$. The motivation for the symplecticity of U comes from the microlocal analysis, it arises naturally when one tries to construct quasimodes for the Laplace operator near $\partial\Omega$ using Fourier integral operators. Indeed, the billiard ball map B can be considered as a boundary map δ_+ for the pair of glancing hypersurfaces $F = T^*_{\partial\Omega}\mathbb{R}^n$, $G = S^*\mathbb{R}^n = \{(x, \xi) \in T^*\mathbb{R}^n ;\ |\xi| = 1\}$. Using Theorem 1 of the present paper, one of the authors proved in [18] that F and G can be put together into the "normal form"

$$F = \{x_n = 0\}, \qquad G = \{\xi_n^2 - x_n - K(I) = x_n p(\varphi, I, \xi_n)\}$$

via an exact symplectic transformation $\chi\colon T^*\mathbb{R}^n \to T^*(\mathbb{T}^{n-1} \times \mathbb{R}^1)$ near $\widetilde{\mathscr{O}} \subset T^*_{\partial\Omega}\mathbb{R}^n$ where $p = 0$ whenever $I \in E$. This is the crucial point used in [18] for the construction of quasimodes for the Laplace operator with Dirichlet (Neumann) boundary conditions whose "frequency set" is just the union of the broken bicharacteristics passing over the invariant tori of the billiard ball map.

In the case $n = 2$ Lazutkin [12] proved the existence of invariant curves Λ_ω for the billiard ball map B near any of the two connected components of $\partial\Sigma = S^*_+\partial\Omega \cup S^*_-\partial\Omega$, $S^*_\pm\partial\Omega = \partial\Omega \times \{\pm 1\}$, with rotation numbers ω belonging to a Cantor set R with a positive measure. Moreover, he constructed a diffeomorphism $U\colon \mathbb{T}^1 \times (\delta_1, \delta_2) \to \Sigma$, $0 < \delta_1 < \delta_2$, mapping $\mathbb{T}^1 \times \{\omega\}$, $\omega \in R$, into Λ_ω and such that $B_0 = U^{-1}BU$ is given by

$$(y, \omega) \to (y + \omega \ (\mathrm{mod}\, 2\pi), \omega) \quad \text{on } \mathbb{T}^1 \times \{\omega\}.$$

In the present paper we give a more precise symplectic version of Lazutkin's result (see Theorem 2) which provides a symplectic "normal form" of B on the invariant curves in a neighbourhood of $\partial\Sigma$. As a consequence we show that the lengths of the periodic broken geodesics in $\overline{\Omega} \subset \mathbb{R}^2$ determine uniquely the invariant curves Λ_ω and $B|_{\Lambda_\omega}$ up to a symplectic map.

Let us expose this result in more details. With any periodic point $\rho \in \Sigma \setminus \partial\Sigma$ of the billiard ball map one can associate two integers n, $m \in \mathbb{N}$ where n is the period and m is the winding number normalized by $2m \leq n$. Denote by $\Gamma(m, n)$ the set of periodic orbits $\mathfrak{g} = \{g_1, \dots, g_n\}$, $Bg_j = g_{j+1 \ (\mathrm{mod}\ n)}$ with a winding number m.

Let $L(m, n)$ be the set of the lengths of the periodic broken geodesics in Ω corresponding to the periodic orbits $\mathfrak{g} \in \Gamma(m, n)$ and denote by

$$\mathfrak{L}(\Omega) = \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{[n/2]} L(m, n) \cup \{nl_0\} \right)$$

the length spectrum of Ω where l_0 is the length of $\partial\Omega$.

Let Ω_1 and Ω_2 be two strictly convex domains in $\mathbb{R}^2$. Denote by B_1 and B_2 the corresponding billiard ball maps acting in Σ^1 and Σ^2 respectively. Guillemin and Melrose conjectured in [6] that if $\mathfrak{L}(\Omega_1) = \mathfrak{L}(\Omega_2)$ and the eigenvalues of the linear parts of the Poincaré maps corresponding to broken geodesics in $\overline{\Omega}_1$ and $\overline{\Omega}_2$ with one and the same length coincide, then B_1 and B_2 can be conjugated by a symplectic map.

In the present paper we give a partial answer to this question. We prove that if $L_1(m, n) = L_2(m, n)$ for $m/n < \delta$, $\delta > 0$, there exists an exact symplectic map $\chi: \Sigma^2 \to \Sigma^1$ and some sets $\Sigma_R^i \subset \Sigma^i$, $i = 1, 2$, of positive measure (see conditions (i), (ii) in §2) and consisting of invariant curves for B_1 and B_2 respectively so that

$$\chi(\Sigma_R^2) = \Sigma_R^1 \quad \text{and} \quad \chi^*(B_1|_{\Sigma_R^1}) = B_2|_{\Sigma_R^2}.$$

We turn now to an outline of the paper.

In §2 the main results are formulated. §3 has a preliminary character. Here we give some facts about the so-called approximate interpolating Hamiltonian introduced by Marvizi and Melrose [14]. This is a C^∞ function ζ defined in a neighborhood of $\widetilde{\mathscr{O}}$ in $T^*\partial\Omega$ which defines $S^*\partial\Omega$ near $\widetilde{\mathscr{O}}$ as $\{\zeta = 0\}$ and such that

$$B(\rho) = \exp(-\zeta^{1/2} X_\zeta)(\rho) + O(\zeta^\infty).$$

§4 is devoted to the construction of a completely integrable Hamiltonian ζ_0 close to ζ and of some "action-angle" coordinates for ζ_0. More precisely, making use of the normal form of Birkhoff for P, we find some symplectic coordinates $(\varphi, I) \in \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$ in which the interpolating Hamiltonian $\zeta(\varphi, I)$ can be regarded as a perturbation of a polynomial $\zeta_0(I)$. The coefficients of $\zeta_0(I)$ depend only on the normal form of Birkhoff and the length of the closed trajectories of X_ζ on the orbit cylinder associated with the nondegenerate trajectory $\widetilde{\mathscr{O}}$. Now the billiard ball map can be regarded as a perturbation of $B_0 = \exp(-X_{\frac{2}{3}\zeta_0^{3/2}})$.

In §5 we apply the Kolmogorov-Arnold-Moser (KAM) theory to the map B_0 close to B. For this purpose, using some ideas of [4, 5], we reduce the problem to finding invariant tori for a suitable Hamiltonian system close to a completely integrable one. The respective Hamiltonian is nondegenerate in the interior of Σ but it has a singularity of the form $\zeta_0^{3/2}$ on $\partial\Sigma$. To overcome this difficulty we first consider the corresponding Hamiltonian systems in some compacts a way from $\partial\Sigma$ and apply a refined version of Pöschel's theorem [19] following the dependence on the various constants (see the Appendix). Next we glue the symplectic maps obtained together using some uniqueness results about the invariant tori. In §6 we consider the case $n = 2$.

2. Main results

First let us recall the definition of the billiard ball map $B\colon \Sigma \to \Sigma$ (cf. [6]). Denote by $\nu(x)$ the exterior normal vector at $x \in \partial\Omega$ normed by $|\nu(x)| = 1$. If $\xi \in T_x^*\partial\Omega$ and $|\xi| < 1$, then there exists a unique vector $e(x,\xi) \in \mathbb{R}^n$ such that $|e(x,\xi)| = 1$, $\langle \nu(x), e(x,\xi)\rangle < 0$ and $\langle v,\xi\rangle = \langle v, e(x,\xi)\rangle$ for any $v \in T_x\partial\Omega$. Here $\langle\, ,\, \rangle$ denotes the scalar product in $\mathbb{R}^n$ and $T_x^*\partial\Omega$ and $T_x\partial\Omega$ are identified with $\mathbb{R}^{n-1}$ via the Euclidean metric. Denote by y the first point of intersection of the ray $\{x + te(x,\xi),\ t > 0\}$ with $\partial\Omega$. This point is unique if the hypersurface is strictly convex. Let $\eta \in T_y^*\partial\Omega$ be such that $|\eta| \le 1$ and $\langle v,\ e(x,\xi)\rangle = \langle v,\ \eta\rangle$ for any $v \in T_y\partial\Omega$. For $(x,\xi) \in \Sigma \cap \{|\xi| < 1\}$ we define $B(x,\xi) = (y,\eta)$ and extend B to $\partial\Sigma$ by $B(x,\xi) = (x,\xi)$ for $|\xi| = 1$.

Let us give a precise formulation of the assumptions imposed on the closed geodesic $\mathscr{O}$ on the C^∞ hypersurface $\partial\Omega$. Let ρ be an arbitrary point of $\widetilde{\mathscr{O}}$ and let $W \subset \partial\Sigma$ be a local transversal section of $\widetilde{\mathscr{O}}$ at ρ of dimension $2n-4$. Denote by P the Poincaré map associated with the Hamiltonian flow of X_H, $H(x,\xi) = |\xi|$, $(x,\xi) \in T^*\partial\Omega$ and by dP the differential of P at ρ. First we assume that $\mathscr{O}$ is elliptic, i.e. all the eigenvalues λ_k, λ_k^{-1}, $k = 1,2,\dots,n-2$, of dP are on the unit circle $\{\lambda \in \mathbb{C}; |\lambda| = 1\}$ and $\lambda_k \ne \pm 1$. Next we suppose that the Poincaré map P is $2N+1$-elementary for some half-integer $N \ge 3/2$. This means that the eigenvalues of dP are distinct and

$$\prod_{k=1}^{n-2} \lambda_k^{j_k} \ne 1 \quad \text{if } 1 \le \sum_{k=1}^{n-2} |j_k| \le 2N+1. \tag{2.1}$$

Let ω be the canonic symplectic form on $T^*\partial\Omega$. There exist local coordinates (z,ζ) mapping a neighbourhood of ρ in W into a neighbourhood of $O \in \mathbb{R}^{2n-4}$ such that $\omega|_W = \sum_{j=1}^{n-2} dz_j \wedge d\zeta_j$ and if we introduce polar coordinates (φ, r) by

$$z_j = \sqrt{2r_j}\cos\varphi_j, \quad \zeta_j = \sqrt{2r_j}\sin\varphi_j, \quad j = 1,2,\dots,n-2,$$

then we can represent the Poincaré map $P\colon W \to W$ as

$$P(\varphi, r) = (\varphi + B_0 + B_1 r + \cdots + O(|r|^N), r + O(|r|^{N+1})). \tag{2.2}$$

Form (2.2) is called normal form of Birkhoff (cf. [9]). We make the nondegeneracy assumption

$$\det B_1 \ne 0. \tag{2.3}$$

In order to formulate our main results we shall need some additional notions.

A C^1 diffeomorphism from T^*M into T^*N where M and N are C^∞ manifolds is said to be *exact symplectic* if it preserves the integrals of the fundamental 1-form over the fundamental cycles.

A function $S(y,\xi)$ is said to *generate* an exact symplectic transformation $T: T^*M \to T^*M$, where M is either $\mathbb{R}^n$ or $\mathbb{T}^n$ if

$$\text{graph } T = \{(x,\xi; T(x,\xi)); (x,\xi) \in T^*M\}$$

has the form

$$\text{graph } T = \{(y - S_\xi(y,\xi), \xi; y, \xi - S_y(y,\xi))\}$$

and $|\det S_{y\xi}(y,\xi)| < 1$ for $(y,\xi) \in T^*M$.

Denote by $\Gamma \subset \mathbb{R}^{n-1}$ a set of the form

$$\begin{aligned}(2.4)\quad \Gamma = \{I \in \mathbb{R}^{n-1}; C_1 I_1 \le I_j \le C_2 I_1, j = 2, \dots, n-2,\\ C_3 I_1^{2b} \le t_0 - I_{n-1} \le C_4 I_1^{2b}, 0 < I_1 < a_0\}\end{aligned}$$

where C_j, a_0 and $b < 1/2$ are some positive constants and $2\pi t_0$ is the period of $\widetilde{\mathscr{O}}$.

For any point $(p, q, \varphi_{n-1}, I_{n-1}) \in \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times T^1 \times \mathbb{R}^1$ we introduce *symplectic polar* coordinates $(\varphi, I) \in \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$ by

$$p_j = \sqrt{2I_j}\cos\varphi_j, \quad q_j = \sqrt{2I_j}\sin\varphi_j, \qquad j = 1, \dots, n-2.$$

Denote by V the set of points $(p, q, \varphi_{n-1}, I_{n-1})$ with symplectic polar coordinates $(\varphi, I) \in \mathbb{A}^{n-1} = \mathbb{T}^{n-1} \times \Gamma$ and let $\widetilde{\mathscr{O}}^0 = \{(0, 0, \varphi_{n-1}, t_0); \varphi_{n-1} \in \mathbb{T}^1\} \subset \overline{V}$, $\overline{V}$ being the closure of V.

Our main result is

Theorem 1. *Let Ω be strictly convex in a neighbourhood of a closed elliptic geodesic $\mathscr{O} \subset \partial\Omega$ such that the corresponding Poincaré map P satisfies* (2.1) *with $N \ge 2$ and* (2.3). *Then there exists an exact symplectic diffeomorphism $U: \overline{V} \to U(\overline{V}) \subset \Sigma$, $U(\widetilde{\mathscr{O}}^0) = \widetilde{\mathscr{O}} = U(\overline{V}) \cap \partial\Sigma$, a Cantor set $E \subset \overline{\Gamma}$ with a positive Lebesgue measure, $(0, t_0) \in E$, some smooth functions K and g in $\overline{\Gamma}$ and $\overline{\mathbb{A}^{n-1}}$ respectively, $K(0, t_0) = 0$, $\operatorname{grad} K(0, t_0) \ne 0$, so that in polar coordinates $(\varphi, I) \in \mathbb{A}^{n-1}$ the exact symplectic map $B_0(\varphi, I) = U^{-1}BU(\varphi, I)$ is generated by the function $-\frac{2}{3}(K(I))^{3/2} + g(\varphi, I)$ and*

$$(2.5)\qquad g(\varphi, I) = 0 \quad \textit{for any } \varphi \in \mathbb{T}^{n-1},\ I \in E.$$

Here $K \in C^\infty(\Gamma) \cap C^\kappa(\overline{\Gamma})$, $D_\varphi^\alpha g \in C^\infty(\mathbb{A}^{n-1}) \cap C^\kappa(\overline{\mathbb{A}^{n-1}})$ for any multi-index α and $U \in C^\infty(V) \cap C^{\kappa-1}(\overline{V})$ where κ is the entire part of $N - 1$. Moreover, if $N \ge 3$, the set E can be chosen so that

$$(2.6)\qquad \operatorname{mes}(E \cap B_a)/\operatorname{mes}(\Gamma \cap B_a) = 1 - O(a^b)$$

where B_a is the ball of radius $a \le a_0$.

Remark 2.1. Equality (2.5) yields $D_\varphi^\alpha D_I^\beta g(\varphi, I) = 0$ for any $\varphi \in \mathbb{T}^{n-1}$, $I \in E$ since E has no isolated points. Thus (2.5) is equivalent to

$$B_0(\varphi, I) = (\varphi + \operatorname{grad} \tau(I) \pmod{2\pi}, I) \quad \text{for } (\varphi, I) \in \mathbb{T}^{n-1} \times E$$

where $\tau(I) = -\frac{2}{3}(K(I))^{3/2}$. Moreover, $D_I^\beta g(\varphi, 0) = 0$ for any β, $|\beta| \le \kappa$.

Let us turn now to the case $n = 2$.

Suppose that Ω is a strictly convex domain in $\mathbb{R}^2$ with a C^∞ smooth boundary $\partial\Omega$ of length $l_0 = 2\pi\mathfrak{l}$. Denote $\Gamma = [\mathfrak{l} - \delta_0, \mathfrak{l}] \subset \mathbb{R}^1$, $\mathbb{A} = \mathbb{T}^1 \times \Gamma$ for some positive constant δ_0. Then the following theorem is valid which is a counterpart of Theorem 1 for $n = 2$.

Theorem 2. *Let Ω be a strictly convex planar domain with a smooth boundary of length l_0. Then there exists an exact symplectic diffeomorphism $U: \mathbb{A} \to \Sigma$, $U(\mathbb{T}^1 \times \{\mathfrak{l}\}) = S_+^*\partial\Omega$, a Cantor set $E \subset \Gamma$ with a positive Lebesgue measure, $\mathfrak{l} \in E$, and some C^∞ functions $K(I)$ and $g(\varphi, I)$ in Γ and $\mathbb{A}$ respectively, $K(\mathfrak{l}) = 0$, $K'(\mathfrak{l}) < 0$, such that the exact symplectic map $B_0 = U^{-1}BU$ is generated in $\mathbb{A}$ by the function $-\frac{2}{3}(K(I))^{3/2} + g(\varphi, I)$ and (2.5) is satisfied for any $\varphi \in \mathbb{T}^1$, $I \in E$.*

Moreover, the set E can be chosen so that for any $\delta \in (0, \delta_0]$

$$\delta - \operatorname{mes}(E \cap [\mathfrak{l} - \delta, \mathfrak{l}]) \le C_n \delta^N \tag{2.7}$$

Remark 2.2. Equality (2.5) for $n = 2$ yields again

$$B_0(\varphi, I) = (\varphi + \tau'(I), I) + Q(\varphi, I)$$

where $\tau(I) = -\frac{2}{3}(K(I))^{3/2}$ and $Q(\varphi, I) = 0$ for any $(\varphi, I) \in \mathbb{T}^1 \times E$.

Remark 2.3. The invariant tori given in Theorem 2 can be enumerated by their rotation numbers which in the two-dimensional case determine the invariant tori uniquely. The set R of rotation numbers is defined by the small denominator condition as follows. First for any $j \in \mathbb{N}$ consider

$$R_j = \{\omega \in (4^{-j}, 4^{-j+1}); |\omega k_1 - k_2| \ge \mu 2^{-j^2-2j}|k_1|^{-\sigma} \quad \text{for any } (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}\} \tag{2.8}$$

where $\sigma > 1$ and $\mu > 0$ do not depend on j. Now, we write $R = (\bigcup_{j=1}^\infty R_j) \cap (0, \delta_0)$ for some $\delta_0 > 0$. Then the set E can be given by $E = \{I \in \Gamma; \tau'(I) \in \overline{R}\}$ as we shall see in the proof of Theorem 2.

Denote $\Sigma_R = U(\mathbb{T}^1 \times E)$. Obviously, Σ_R is a subset of Σ such that

(i) Σ_R is the union of invariant curves of B with rotation numbers in R.

(ii) $\operatorname{mes}(\Sigma^\delta) - \operatorname{mes}(\Sigma^\delta \cap \Sigma_R) \le C_N \delta^N$ for any $\delta \in (0, \delta_0]$, $N > 0$ and some $C_N > 0$ where $\Sigma^\delta = \partial\Omega \times [1 - \delta, 1]$.

Let Ω_1 and Ω_2 be two strictly convex domains in $\mathbb{R}^2$. Denote by B_1 and B_2 the corresponding billiard ball maps acting in Σ^1 and Σ^2 respectively and by $L_i(m,n)$ the sets $L(m,n)$ defined as in §1 for Ω_i, $i=1,2$. Let R be given as in Remark 2.3, and Σ_R^i, $i=1,2$, be the union of the invariant curves of B_i, $i=1,2$, with rotation numbers in R.

Theorem 3. *Suppose that* $L_1(m,n)=L_2(m,n)$ *if* $m/n<\delta$ *for some* $\delta>0$. *Then there exists an exact symplectic map* $\chi\colon \Sigma^2\to\Sigma^1$ *such that* $\chi(\Sigma_R^2)=\Sigma_R^1$ *and* $\chi^*(B_1|_{\Sigma_R^1})=B_2|_{\Sigma_R^2}$. *Moreover, the set* E *and the restriction of* K *on* E *are determined uniquely by the set of rotation numbers* R *and by* $L(m,n)$ *for* $m/n<\delta$.

Theorem 1 was announced in a slightly weaker form in [11] and Theorems 2 and 3 in [17] where an idea of the proof was also given.

3. Interpolating Hamiltonian

The equivalence theorem of Melrose [15] for nondegenerate glancing points (cf. also [8]) can be applied to the transversally intersecting hypersurfaces $S^*\mathbb{R}^n = \{(x,\xi)\in T^*\mathbb{R}^n; |\xi|=1\}$ and $T^*_{\partial\Omega}\mathbb{R}^n=\{(x,\xi)\in T^*\mathbb{R}^n; x\in\partial\Omega\}$ in the symplectic manifold $T^*\mathbb{R}^n$ since the manifold of glancing points $\partial\Sigma=S^*\partial\Omega$ consists entirely of nondegenerate glancing points provided that Ω is strictly convex in a neighbourhood of $\mathscr{O}$ (see [13]). Thus in a neighbourhood $U\subset T^*\partial\Omega$ of any point $(x,\xi)\in\widetilde{\mathscr{O}}\subset\partial\Sigma$ we can introduce symplectic coordinates $(y,\eta)=(y',y_{n-1},\eta',\eta_{n-1})=\chi(x,\xi)$ such that $\partial\Sigma\cap U=\{\eta_{n-1}=0\}$, $\eta_{n-1}\geq 0$ in $\Sigma\cap U$ and

$$\chi\circ B\circ\chi^{-1}(y,\eta)=(y',y_{n-1}-\eta_{n-1}^{1/2},\eta).$$

Then

$$B(x,\xi)=\exp(-\eta_{n-1}^{1/2}(x,\xi)X_{\eta_{n-1}})(x,\xi).$$

We can say that B is locally interpolated by the Hamiltonian flow generated by the function $\eta_{n-1}(x,\xi)$ which is called a local interpolating Hamiltonian for the billiard ball map.

We shall use the following proposition whose assertion is given in [13] without proof.

Proposition 3.1. *Let* ζ_j, $j=1,2$, *be local interpolating Hamiltonians for the billiard ball map in a neighbourhood* U *of* $\rho_0\in\partial\Sigma$. *Then* $\zeta_1-\zeta_2$ *vanishes to infinite order on* $\partial\Sigma$.

Proof. As noted in [13], the assertion of Proposition 3.1 can be derived from the proof of Melrose [15]. For the sake of completeness here we give a direct proof of it which is close to the proof in the two-dimensional case [14].

Let (x,ξ) and (y,η) be symplectic coordinates in a neighbourhood U of $\rho_0\in\partial\Sigma$ such that $x(\rho_0)=\xi(\rho_0)=y(\rho_0)=\eta(\rho_0)=0$, $U\cap\partial\Sigma=\{\xi_{n-1}=0\}=\{\eta_{n-1}=0\}$, $\xi_{n-1}\geq 0$ and $\eta_{n-1}\geq 0$ in $U\cap\Sigma$ and

$$B(x,\xi)=(x',x_{n-1}-\xi_{n-1}^{1/2},\xi),\qquad B(y,\eta)=(y',y_{n-1}-\eta_{n-1}^{1/2},\eta) \tag{3.1}$$

in $U \cap \Sigma$. Here $\xi_{n-1} = \zeta_1$, $\eta_{n-1} = \zeta_2$ are the respective local interpolating Hamiltonians. Denote by $x = x(y, \eta)$, $\xi = \xi(y, \eta)$ the symplectic change of variables $(y, \eta) \to (x, \xi)$ defined in a neighbourhood U_0 of $(0, 0)$. From (3.1) we have

$$\begin{aligned} &\xi_j(y, \eta) = \xi_j(y', y_{n-1} - \eta_{n-1}^{1/2}, \eta), \qquad j = 1, \dots, n-1, \\ &x_j(y, \eta) = x_j(y', y_{n-1} - \eta_{n-1}^{1/2}, \eta), \qquad j = 1, \dots, n-2, \\ &x_{n-1}(y, \eta) - \xi_{n-1}^{1/2}(y, \eta) = x_{n-1}(y', y_{n-1} - \eta_{n-1}^{1/2}, \eta) \end{aligned} \tag{3.2}$$

in $U_0 \cap \{\eta_{n-1} \geq 0\}$ if U_0 is small enough.

Now we shall make use of the following lemma.

Lemma 3.2. *Let $p(z, \zeta)$ be a C^∞ function which satisfies for all $\zeta \geq 0$ the equality*

$$p(z - \zeta^{1/2}, \zeta) = p(z, \zeta) + O(\zeta^\infty). \tag{3.3}$$

Then we have $(\partial p/\partial z)(z, \zeta) = O(\zeta^\infty)$.

To prove the lemma, it suffices to show that

$$\frac{\partial^{n+1} p}{\partial z \partial \zeta^n}(z, 0) = 0$$

for any nonnegative integer n. This equality is proved by induction using Taylor's series of both sides of (3.3) around $(z, 0)$.

From the first $2n - 3$ equalities of (3.2) we obtain

$$\begin{aligned} &\{\eta_{n-1}, \xi_j\} = \frac{\partial \xi_j}{\partial y_{n-1}}(y, \eta) = O(\eta_{n-1}^\infty), \qquad j = 1, \dots, n-1, \\ &\{\eta_{n-1}, x_j\} = \frac{\partial x_j}{\partial y_{n-1}}(y, \eta) = O(\eta_{n-1}^\infty), \qquad j = 1, \dots, n-2, \end{aligned}$$

because $\xi_j(y, \eta)$, $j = 1, \dots, n-1$, and $x_j(y, \eta)$, $j = 1, \dots, n-2$, are smooth functions of (y_{n-1}, η_{n-1}) depending on the parameters (y', η') and we can apply Lemma 3.2. Here $\{\ ,\ \}$ stands for Poisson's bracket in $T^*\mathbb{R}^{n-1}$ associated with the standard symplectic form.

If we consider, conversely, η_{n-1} as a function of (x, ξ), we find

$$\begin{aligned} &\frac{\partial \eta_{n-1}}{\partial x_j} = -\{\eta_{n-1}, \xi_j\} = O(\eta_{n-1}^\infty), \qquad j = 1, \dots, n-1, \\ &\frac{\partial \eta_{n-1}}{\partial \xi_j} = \{\eta_{n-1}, x_j\} = O(\eta_{n-1}^\infty), \qquad j = 1, \dots, n-2, \end{aligned}$$

hence $\eta_{n-1} = \eta_{n-1}(\xi_{n-1}) + O(\xi_{n-1}^\infty)$ and $\partial \eta_{n-1}(0)/\partial \xi_{n-1} > 0$ since the change is nonsingular and in $U \cap \Sigma$ both η_{n-1} and ξ_{n-1} are nonnegative. Thus $\xi_{n-1} =$

$f(\eta_{n-1})+O(\eta_{n-1}^{\infty})$ where f is a smooth function such that $f(0)=0$, $f'(0)>0$ and $f(\eta_{n-1})>0$ for $\eta_{n-1}>0$. From the last equality of (3.2) we have

$$x_{n-1}(y,\eta)-f^{1/2}(\eta_{n-1})=x_{n-1}(y',y_{n-1}-\eta_{n-1}^{1/2},\eta)+O(\eta_{n-1}^{\infty}), \tag{3.4}$$

whence we obtain

$$\frac{\partial x_{n-1}}{\partial y_{n-1}}(y,\eta)=\frac{\partial x_{n-1}}{\partial y_{n-1}}(y',y_{n-1}-\eta_{n-1}^{1/2},\eta)+O(\eta_{n-1}^{\infty}).$$

Now Lemma 3.2 yields

$$\frac{\partial^2 x_{n-1}}{\partial^2 y_{n-1}}(y,\eta)=O(\eta_{n-1}^{\infty}),$$

hence

$$x_{n-1}(y,\eta)=a(y',\eta)y_{n-1}+b(y',\eta)+O(\eta_{n-1}^{\infty})$$

where $a(y',\eta)$ and $b(y',\eta)$ are C^{∞} functions. Then from (3.4) we obtain $a(y',\eta)=(f(\eta_{n-1})/\eta_{n-1})^{1/2}+O(\eta_{n-1}^{\infty})$ for $\eta_{n-1}>0$. Now, as in [14] we have

$$\begin{aligned}1&=\{\xi_{n-1},x_{n-1}\}=\{f(\eta_{n-1}),(f(\eta_{n-1})/\eta_{n-1})^{1/2}y_{n-1}+b(y',\eta)\}+O(\eta_{n-1}^{\infty})\\&=f'(\eta_{n-1})(f(\eta_{n-1})/\eta_{n-1})^{1/2}+O(\eta_{n-1}^{\infty})\quad\text{for } \eta_{n-1}>0,\end{aligned}$$

hence

$$f^{3/2}(\eta_{n-1})=\eta_{n-1}^{3/2}+C+O(\eta_{n-1}^{\infty}).$$

Since $f(0)=0$, then $C=0$, i.e. $\xi_{n-1}=\eta_{n-1}+O(\eta_{n-1}^{\infty})$ and Proposition 3.1 is proved.

Let $\{\varphi_j\}_{j=1}^{J}$ be a smooth partition of unity in a neighbourhood U of $\tilde{\mathscr{O}}$ in $T^*\partial\Omega$ and let ζ_j be local interpolating Hamiltonians for the billiard ball map defined respectively in a neighbourhood of $\operatorname{supp}\varphi_j$. Now define $\zeta=\sum_{j=1}^{J}\varphi_j\zeta_j$. From Proposition 3.1 it follows that $\zeta-\zeta_j$ vanishes to infinite order on $\partial\Sigma$. Now we have $B(\rho)=\exp(-\zeta^{1/2}X_{\zeta})(\rho)+O(\zeta^{\infty}(\rho))$ in a neighbourhood of $\tilde{\mathscr{O}}$. Moreover, $\zeta=0$ defines $\partial\Sigma$ in a neighbourhood of $\tilde{\mathscr{O}}$ and the bicharacteristic lines passing through the points $\rho\in\partial\Sigma$ coincide with $\{\exp(tX_{\zeta})(\rho);\zeta=0,\ t\in\mathbb{R}^1\}$.

The function ζ is called an approximate interpolating Hamiltonian. In the case $n=2$ its construction is carried out directly in [14] and its uniqueness in the sense of Proposition 3.1 is proved.

4. Construction of suitable "action-angle" coordinates

4.1. Let S be a transversal to $\tilde{\mathscr{O}}$ in Σ and $S_h=\{\rho\in S;\zeta(\rho)=h\}$, $S_0=W$. Since 1 is not an eigenvalue of the differential dP of the Poincaré map P at $\rho\in S_0\cap\tilde{\mathscr{O}}$, there exists a smooth family of closed trajectories $\tilde{\mathscr{O}}_h$ of X_{ζ} contained respectively in the sets $\{\rho\in\Sigma;\zeta(\rho)=h\}$ for $|h|$ small which form an orbit cylinder and $\tilde{\mathscr{O}}_0=\tilde{\mathscr{O}}$.

We shall use the normal form of Birkhoff for the Poincaré map $P_{\zeta}: S_{\zeta}\to S_{\zeta}$ depending smoothly on the parameter ζ.

Proposition 4.1. *Let the Poincaré map be $2N+1$-elementary, $N \geq 3/2$. Then there exist symplectic coordinates (p, q, ζ, t) in a neighbourhood of S, i.e. $\omega = \sum_{j=1}^{n-2} dp_j \wedge dq_j + d\zeta \wedge dt$ such that $S \subset \{t = 0\}$ and if we denote $P_\zeta(p, q) = (p^*, q^*)$, then*

$$\begin{aligned} p_j^* &= p_j \cos\Phi_j(r', \zeta) - q_j \sin\Phi_j(r', \zeta) + f_j(p, q, \zeta), \\ q_j^* &= p_j \sin\Phi_j(r', \zeta) + q_j \cos\Phi_j(r', \zeta) + f_{n-2+j}(p, q, \zeta), \\ & \qquad j = 1, \dots, n-2, \end{aligned} \tag{4.1}$$

where $r' = (r_1, \dots, r_{n-2})$, $r_j = (p_j^2 + q_j^2)/2$ and

$$|f_j(p, q, \zeta)| \leq C(|p|^2 + |q|^2)^{N+1/2}, \qquad j = 1, \dots, 2n-4. \tag{4.2}$$

Moreover,

$$\Phi_j(r', \zeta) = \frac{\partial A}{\partial r_j}(r', \zeta), \qquad j = 1, \dots, n-2,$$

where A is a polynomial of r' of degree at most N with C^∞ coefficients and $\rho_0^h(0, 0, h)$ lie on the cylinder of orbits of X_ζ. We choose $A(r', \zeta)$ so that $A(0, \zeta) = 0$.

The proof of Proposition 4.1 is analogous to that given in [9, 16], constructing a generating function of the respective canonical transformation depending smoothly on ζ. Thus we construct the coordinates p and q in S. Then we extend p and q to a neighbourhood of S so that $X_\zeta p_j = X_\zeta q_j = 0$, $j = 1, \dots, n-2$, and supplement p, q, ζ to a symplectic basis in a neighbourhood of S. It is easily seen that in this basis

$$X_\zeta = \partial/\partial t. \tag{4.3}$$

Let $2\pi t_0$ be a period of $\widetilde{\mathscr{O}}$. Denote by $\widetilde{G}\colon U_0 \to \widetilde{G}(U_0) \subset \mathbb{R}^{2n-2}$ the symplectic map $\widetilde{G}(\rho) = (p, q, t/t_0, t_0\zeta)$ defined in a neighbourhood U_0 of S and set $G^0 = \widetilde{G}|_{S_\zeta}$.

For δ small enough we denote

$$\begin{aligned} V_\delta = \{(p, q, \varphi_{n-1}, I_{n-1}) \in \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{T}^1 \times [t_0 - \delta, t_0 + \delta]; \\ p_j^2 + q_j^2 \leq 2\delta, j = 1, \dots, n-2\} \end{aligned}$$

with a symplectic form $\tilde{\omega} = dp \wedge dq + dI_{n-1} \wedge d\varphi_{n-1}$. We shall denote by

$$(\varphi, I) = \pi(p, q, \varphi_{n-1}, I_{n-1}) \in \Pi^{n-1} = \mathbb{T}^{n-1} \times (0, \delta]^{n-2} \times [t_0 - \delta, t_0 + \delta]$$

the respective polar coordinates in V_δ determined by

$$p_j = \sqrt{2I_j}\cos\varphi_j, \quad q_j = \sqrt{2I_j}\sin\varphi_j, \qquad j = 1, \dots, n-2.$$

Note that for $\tilde{\omega}^0 = \sum_{j=1}^{n-1} dI_j \wedge d\varphi_j$ we have $\pi^*\tilde{\omega}^0 = \tilde{\omega}$ in Π^{n-1}.

4.2. Let $l(\zeta)$ be the period of periodic trajectory $\tilde{\mathscr{O}}_\zeta$ from the cylinder of orbits, thus $l(0) = 2\pi t_0$. Denote by $l^0(\zeta)$ and $A^0(r', \zeta)$ Taylor's expansions of $l(\zeta)$ and $A(r, \zeta)$ with respect to ζ at $\zeta = 0$ up to degrees M and $M+1$ respectively, M sufficiently large. Let ζ_0 be the solution of the equation

$$(4.4) \qquad 2\pi t_0 - \int_0^\zeta l^0(h)\,dh - A^0(r', \zeta) = 2\pi I_{n-1}, \qquad \zeta(0, t_0) = 0,$$

for $(r', t_0 - I_{n-1}) \in [-\delta, \delta]^{n-1}$. Then we have

$$(4.5) \qquad \frac{\partial \zeta_0(0, t_0)}{\partial I_{n-1}} < 0.$$

The following proposition provides "action-angle" coordinates for a completely integrable Hamiltonian ζ_0 which is close to ζ.

Proposition 4.2. *For δ small enough there exists a neighbourhood U_δ of $\tilde{\mathscr{O}}$ in $T^*\partial\Omega$ and an exact symplectic transformation $G\colon V_\delta \to U_\delta$ of class C^∞ such that $G^*\zeta = \zeta \circ G$ has the form*

$$G^*\zeta(p, q, \varphi_{n-1}, I_{n-1}) = \zeta_0(I) + \tilde{\zeta}(p, q, \varphi_{n-1}, I_{n-1})$$

and

$$(4.6) \qquad |\tilde{\zeta}(p, q, \varphi_{n-1}, I_{n-1})| \le C(|I'|^{N+1/2} + |\zeta_0|^{M+2})$$

where C does not depend on δ, $I' = (I_1, \dots, I_{n-2})$, $I_j = (p_j^2 + q_j^2)/2$, $j = 1, \dots, n-2$, $I = (I', I_{n-1})$.

Proof. First we shall construct suitable smooth coordinates in a neighbourhood U of $\tilde{\mathscr{O}}$.

Denote by g^t the Hamilton flow of $r_{n-1} = t_0\zeta$ and for $\rho \in S_\zeta$, ρ in a neighbourhood of $\rho_0^0 = \tilde{\mathscr{O}} \cap S_0$ denote by $\tau(\rho)$ the smallest positive time such that $g^{\tau(\rho)}(\rho) \in S_\zeta$, $\tau(\rho_0^0) = 2\pi$. Let $s(\rho)$ be the smallest nonnegative time for which $g^{-s(\rho)}(\rho) \in S$ for ρ in a neighbourhood of $\tilde{\mathscr{O}}$.

Before constructing a suitable coordinate system near $\tilde{\mathscr{O}}$ it is convenient to represent the map P_ζ in the form $P_\zeta = P^0 + P^1$ (for the sake of brevity we drop ζ) where $P^0(p, q) = (\tilde{p}, \tilde{q})$,

$$\tilde{p}_j = p_j \cos\Phi_j(r', \zeta) - q_j \sin\Phi_j(r', \zeta),$$

$$\tilde{q}_j = p_j \sin\Phi_j(r', \zeta) + q_j \cos\Phi_j(r', \zeta), \qquad j = 1, \dots, n-2,$$

$$P^1(p, q) = (f_1(p, q, \zeta), \dots, f_{2n-4}(p, q, \zeta)).$$

Obviously, $P^1(\rho) = O(|r'(\rho)|^{N+1/2})$.

Let $\chi \in C^\infty(\mathbb{R}^1)$, $0 \le \chi \le 1$, $\chi'(t) \ge 0$ and $\chi(t) = 0$ in $(-\infty, \varepsilon)$, $\chi(t) = 1$ in $(2\pi - \varepsilon, \infty)$ for some $\varepsilon > 0$ small enough. Denote by $P_t^0\colon \mathbb{R}^{2n-4} \to \mathbb{R}^{2n-4}$ the map $P_t^0(p, q) = (p^*, q^*)$ where

$$p_j^* = p_j \cos(\chi(t)\Phi_j(r', \zeta)) - q_j \sin(\chi(t)\Phi_j(r', \zeta)),$$

$$q_j^* = p_j \sin(\chi(t)\Phi_j(r',\zeta)) + q_j \cos(\chi(t)\Phi_j(r',\zeta)), \qquad j = 1, \dots, n-2.$$

Denote by P_t^1 the map $P_t^1(\rho) = \chi(t)P^1(\rho)$ and set $P^t = P_t^0 + P_t^1$. Now consider the map

$$F'(\rho) = (p(\rho), q(\rho)) = P^{s(\rho)} \circ G^0 \circ g^{-s(\rho)}(\rho). \tag{4.7}$$

It is easy to see that F' is a well-defined C^∞ map in a neighbourhood of $\widetilde{\mathscr{O}}$ since $P^t = \mathrm{Id}$ for $t \le \varepsilon$ and $P^t = P_\zeta$ for $t \ge 2\pi - \varepsilon$.

Next we define the smooth function φ_{n-1} as follows. Denote $T(t,\rho) = t + \chi(t)(2\pi - \tau(\rho))$. We set for $\rho \in U$ a neighbourhood of $\widetilde{\mathscr{O}}$, $\varphi_{n-1}(\rho) = T(s(\rho), g^{-s(\rho)}(\rho))$.

It is easy to see that $\varphi_{n-1}\colon U \to \mathbb{T}^1$ is a well-defined C^∞ function. Moreover, the map

$$U \ni \rho \to F(\rho) = (F'(\rho), \varphi_{n-1}(\rho), r_{n-1}(\rho))$$

is a diffeomorphism for U small enough and $F|_{U_0} = \widetilde{G}|_{U_0}$.

Let $(\varphi, r) = \pi(p, q, \varphi_{n-1}, r_{n-1})$ be the respective polar coordinates

$$p_j = \sqrt{2r_j}\cos\varphi_j, \quad q_j = \sqrt{2r_j}\sin\varphi_j, \quad j = 1, \dots, n-2.$$

Denote $r' = (r_1, \dots, r_{n-2})$.

Now in polar coordinates (φ, r) we denote by U_1 the set

$$U_1 = \{\rho \in U; C_1 r_1(\rho) \le r_j(\rho) \le C_2 r_1(\rho), j = 2, \dots, n-2\}$$

where $0 < C_1 < 1 < C_2$ are some constants.

Lemma 4.3. *In U_1 we have*

$$\begin{aligned} &|\{r_j, r_k\}(\rho)| \le C|r'(\rho)|^{N+1}, && j,k \le n-1, \\ &\{r_j, \varphi_k\}(\rho) = \delta_{jk} + O(|r'|^N), && j \le n-2, k \le n-1, \\ &\{r_{n-1}, \varphi_{n-1}\}(\rho) = 1 + O(|r'|). \end{aligned} \tag{4.8}$$

Proof. From (4.1), (4.2), and (4.7) it is easy to see that

$$|(g^t)^* r - r| \le C|r'|^{N+1}, \qquad |t| \le 4\pi. \tag{4.9}$$

Indeed,

$$\begin{aligned}(g^t)^* F'(\rho) &= P^{s(\rho)+t} \circ G^0 \circ g^{-s(\rho)}(\rho) = P^{s(\rho)+t} \circ P^{-s(\rho)} \circ F'(\rho) \\ &= P_0^{s(\rho)+t} \circ P_0^{-s(\rho)} \circ F'(\rho) + O(|r'|^{N+1/2})\end{aligned}$$

and $(g^t)^* r_{n-1} = r_{n-1}$ which imply (4.9). Moreover, from the above equality it follows that for j, $k \le n-2$

$$(g^t)^* \varphi_j(\rho) = (\chi(s(\rho) + t) - \chi(s(\rho)))\Phi_j(r(\rho)) + O(|r'(\rho)|^N). \tag{4.10}$$

Therefore,

$$\begin{aligned} X_{r_{n-1}}\{r_j, \varphi_k\} &= \{r_j, X_{r_{n-1}}\varphi_k\} + O(|r'|^{N+1/2}) \\ &= \{r_j, \chi'(s(\rho))\Phi_k(r)\} + O(|r'|^N) \\ &= \Phi_k(r)\chi''(s(\rho))\{r_j, s(\rho)\} + O(|r'|^N). \end{aligned}$$

On the other hand,

$$X_{r_{n-1}}\{s(\rho), r_j\} = O(|r'|^{N+1/2}).$$

Thus $\{s(\rho), r_j\} = O(|r'|^{N+1/2})$ for $j \le n-2$ and

$$\{r_j, \varphi_k\}(\rho) = \{r_j, \varphi_k\}(g^{-s(\rho)}(\rho)) + O(|r'|^N) = \delta_{jk} + O(|r'|^N).$$

Now we shall prove that

$$\tau(\rho) = f(r(\rho)) + O(|r'|^{N+1/2}), \qquad \rho \in S, \tag{4.11}$$

for some smooth function f such that $f(0) = 2\pi$. Let $\rho^0 \in S$, $\rho = (0, r(\rho))$ in polar coordinates. Denote by g_j^t, $t \in \mathbb{R}^1$, the Hamiltonian flow of r_j, $j = 1, \dots, n-1$. Then $\rho_1 = g_{n-1}^{\tau(\rho^0)}(\rho^0) \in S$ and $r(\rho_1) = r(\rho^0) + O(|r'|^{N+1})$. Now taking $\tau_j(\rho^0) = \Phi_j(r'(\rho^0), \zeta(\rho^0))$, $j = 1, \dots, n-2$, from Proposition 4.1 we have

$$g_1^{\tau_1(\rho^0)} \circ \dots \circ g_{n-1}^{\tau_{n-2}(\rho^0)} g_{n-1}^{\tau(\rho^0)}(\rho^0) = \rho^0 + O(|r'|^{N+1/2}). \tag{4.12}$$

On the other hand,

$$[X_{r_j}, X_{r_k}] = X_{\{r_j, r_k\}} = O(|r'|^{N+1/2})$$

and as in [2] it is easy to see that

$$g_j^{t_j} \circ g_k^{t_k}(\rho) = g_k^{t_k} \circ g_j^{t_j}(\rho) + O(|r'(\rho)|^{N+1/2}). \tag{4.13}$$

Let $\rho \in S$, $r(\rho) = r(\rho^0)$. Then $g_1^{t_1(\rho)} \circ \dots \circ g_{n-2}^{t_{n-2}(\rho)}(\rho^0) = \rho$ for some $t_j(\rho) \in \mathbb{R}^1$ and in view of (4.12), (4.13) we obtain

$$g_1^{\tau_1(\rho^0)} \circ \dots \circ g_{n-2}^{\tau_{n-2}(\rho^0)} \circ g^{\tau(\rho^0)}(\rho) = \rho + O(|r'|^{N+1/2}),$$

thus $g^{\tau(\rho^0)}(\rho) = \rho_1 + O(|r'|^{N+1/2})$ where

$$\rho_1 = g_{n-2}^{-\tau_{n-2}(\rho^0)} \circ \dots \circ g_1^{-\tau_1(\rho^0)}(\rho) \in S$$

or

$$\operatorname{dist}(g^{\tau(\rho^0)}(\rho), S) = O(|r'|^{N+1/2}).$$

Therefore $\tau(\rho) = \tau(\rho^0) + O(|r'|^{N+1/2})$, $\rho^0 = (0, r(\rho))$. Obviously $\tau(\rho^0)$ is a smooth function of $\sqrt{r_1}, \dots, \sqrt{r_{n-2}}$. Let $f(r)$ be its Taylor's series up to order $2N$. Then $f(r)$ satisfies (4.11) and $f(0) = 2\pi$. On the other hand, consider

$\tau(\rho)$ as a function of p, q and denote by $\tau^*(p,q)$ its Taylor series up to order $2N$. Then

$$\tau^*(p,q) = f((p_1^2+q_1^2)/2, \dots, (p_{n-1}^2+q_{n-1}^2)/2)$$

which implies that f is a polynomial of $r_j = (p_j^2+q_j^2)/2$, $j = 1, \dots, n-2$.

The second estimate of (4.8) for $k = n-1$ follows from (4.11) which yields

$$\varphi_{n-1}(\rho) = \widetilde{T}(s(\rho), r(\rho)) + O(|r'|^{N+1/2}).$$

Here

$$\widetilde{T}(t,r) = t + \chi(t)(2\pi - f(r)).$$

Thus $X_{r_j}\varphi_{n-1} = O(|r'|^N)$, $j \le n-2$.

Finally, we have $X_{r_{n-1}}\varphi_{n-1} = 1 + O(|r'|)$ since $f(0) = 2\pi$.

It is easy to see that in polar coordinates (φ, I) we have

$$\pi_* F_* X_{r_j}(\rho) = \frac{\partial}{\partial\varphi_j} + \sum_{k=1}^{n-1}\left(c_{jk}(\rho)\frac{\partial}{\partial r_k} + d_{jk}(\rho)\frac{\partial}{\partial\varphi_k}\right), \qquad j \le n-1,$$

where

$$\begin{aligned} &|c_{jk}(\rho)| \le C|r'|^{N+1}, \quad |d_{jk}(\rho)| \le C|r'|^N, \qquad j \le n-2,\ k \le n-1, \\ (4.14)\quad &|c_{n-1,k}(\rho)| \le C|r'|^{N+1}, \qquad k \le n-1, \\ &|d_{n-1,k}(\rho)| \le C, \quad k \le n-2, \quad |d_{n-1,n-1}(\rho)| \le C|r'|, \end{aligned}$$

uniformly in U_1. Indeed, we have

$$\pi_* F_* X_{r_j}(\rho) = \sum_{k=1}^{n-1}\left(\{r_j, r_k\}(\rho)\frac{\partial}{\partial r_k} + \{r_j, \varphi_k\}(\rho)\frac{\partial}{\partial\varphi_k}\right)$$

and by Lemma 4.3 we prove (4.14).

Let $c \in (0,\delta)^{n-2} \times (-\delta,\delta)$ and $M_c = \{r = c\}$. Then M_c is a compact connected $(n-2)$-dimensional submanifold of V_δ. Denote by ι_c the embedding $M_c \hookrightarrow V_\delta$ and by σ the canonic symplectic 1-form $\sigma = \xi\, dx$ in $T^*\partial\Omega$, $\omega = -d\sigma$.

Lemma 4.4. *There exists $\beta \in \bigwedge^1(U)$ such that $\beta = 0$ in a neighbourhood of S,*

$$(4.15)\qquad \|\beta\|_\rho \le C|r'(\rho)|^{N+1/2}$$

where $\|\cdot\|_\rho$ is the norm in $T_\rho^*(T^*\partial\Omega)$ and for $\sigma_1 = \sigma + \beta$ we have

$$(4.16)\qquad d(\iota_c^*(F^*)^{-1}\sigma_1) = 0 \quad \textit{for any } c \in (0,\delta)^{n-2} \times (-\delta,\delta).$$

Moreover,

$$(4.17)\qquad \{r_j, r_k\}_1 = 0 \quad \textit{in } U$$

where $\{\ ,\ \}_1$ is Poisson's bracket induced by $\omega_1 = -d\sigma_1$.

Proof. We denote

$$Z_j^0 = \frac{\partial}{\partial \varphi_j} + \sum_{k=1}^{n-1} d_{jk}(\rho)\frac{\partial}{\partial \varphi_k}, \quad L_j^0 = \sum_{k=1}^{n-1} c_{jk}(\rho)\frac{\partial}{\partial r_k}, \quad j \le n-1,$$

and $Z_j = \pi_*^{-1} Z_j^0$, $L_j = \pi_*^{-1} L_j^0$. Then the vectors Z_j are tangential to M_c and $|L_j|_\rho \le C|c'|^{N+1/2}$ for any $\rho \in M_c$. Here $|\cdot|_\rho$ is the norm in $T_\rho(T^*\partial\Omega)$. Thus we have

$$\begin{aligned}(4.18)\qquad -d(\iota_c^*(F^*)^{-1}\sigma)(Z_j, Z_k) &= \iota_c^*(F^*)^{-1}\omega(Z_j, Z_k)\\ &= \omega(X_{r_j}, X_{r_k})|_{r=c} + O(|c'|^{N+1}) = O(|c'|^{N+1}).\end{aligned}$$

We write $(F^*)^{-1}\sigma = p\,dq + t\,d\zeta + dQ + \tilde\beta$ where $\tilde\beta$ is a 1-form with C^∞ coefficients. Moreover, from (4.18) we have

$$(4.19)\qquad d(\iota_c^*\tilde\beta)(Z_j, Z_k) = d(\iota_c^*(F^*)^{-1}\sigma)(Z_j, Z_k) + O(|c'|^{N+1}) = O(|c'|^{N+1}).$$

On the other hand,

$$(F^*)^{-1}(\iota_c^*\sigma) = \iota_c^*(p\,dq + dQ + \tilde\beta)$$

where

$$\tilde\beta = \sum_{j=1}^{n-2}(a_j\,dp_j + b_j\,dq_j) + a_0\,d\zeta + b_0\,dt$$

and a_j, b_j are C^∞ functions. We expand a_j, b_j, $j = 1, \dots, n-2$, in Taylor's series up to degree $2N$ with respect to p and q around the point $p = q = 0$ and b_0 up to degree $2N+1$. We obtain $\tilde\beta = \beta_1 + \beta_2$ where $\beta_1 = O(|r'|^{N+1})$ is the remainder in Taylor's formula. Then from (4.19) we have

$$(4.20)\qquad d(\iota_c^*\beta_2)(Z_j, Z_k) = O(|c'|^{N+1}),$$

$$d(\iota_c^*\beta_2) = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} P_{jk}(\varphi', t, \sqrt{c_1}, \dots, \sqrt{c_{n-2}}, c_{n-1})\,d\varphi_j \wedge d\varphi_k$$

where P_{jk} are polynomials of $\sqrt{c_1}, \dots, \sqrt{c_{n-2}}$ of degree at most $2N+1$. In view of (4.20), the definition of Z_j^0 and (4.14) we obtain successively that all coefficients of the polynomials P_{jk} are equal to zero. Hence

$$d(\iota_c^*\beta_2) = 0, \qquad c \in (0, \delta)^{n-2} \times (-\delta, \delta).$$

We denote $\beta = -F^*\beta_1$. Then $\omega_1 = -d\sigma_1$, $\sigma_1 = \sigma + \beta$, is closed on M_c for any c.

Denote by X_f^1 the Hamilton vector field of the function f with respect to the symplectic form ω_1. Let us fix $c \in (0, \delta)^{n-2} \times (-\delta, \delta)$, $p \in M_c$. Then

$$\omega_1(X_{r_j}^1, L) = \langle dr_j, L\rangle = 0, \quad \forall L \in T_pM_c.$$

Since $T_\rho M_c$ is Lagrangian with respect to ω_1, we obtain $X_{r_j}(\rho) \in T_\rho M_c$. Hence

$$\{r_j, r_k\}_1(\rho) = \omega_1(X^1_{r_j}, X^1_{r_k}) = 0$$

which proves the lemma.

Let $\rho \in U$. As in [2], consider the map

$$\mathbb{R}^{n-1} \ni t = (t_1, \dots, t_{n-1}) \to T^t(\rho) = F_1^{t_1} \circ \cdots \circ F_{n-1}^{t_{n-1}}(\rho) \in U$$

where $F_j^{t_j}$ is the flow of the Hamilton vector field $X^1_{r_j}$. Let us note that $F_j^{t_j}$ commute with one another in view of (4.17). Since M_c is compact, the stationary group of $t \to F^t(\rho)$ has $n-1$ generators $e_1, \dots, e_{n-1}$ which can be found explicitly in our case. In a neighbourhood U_0 of S we have $\omega_1 = \omega$, hence $X^1_{r_j} = \partial/\partial\varphi_j$ where $(\varphi, r) = \pi(p, q, \varphi_{n-1}, r_{n-1})$ are the respective polar coordinates. Then in U_0 we have for $j \le n-2$

$$F_j^{t_j}(\varphi, r) = (\varphi_1, \dots, \varphi_{j-1}, \varphi_j + t_j, \varphi_{j+1}, \dots, \varphi_{n-1}, r).$$

In particular, $F_j^{t_j} : S \to S$ for $j \le n-2$ and

$$e_j = (\underbrace{0, \dots, 0}_{j-1 \text{ times}}, 2\pi, \underbrace{0, \dots, 0}_{n-j-1 \text{ times}}), \qquad j \le n-2,$$

is a stationary point for $F^t(\rho)$. Let $t_{n-1}(r)$ be the smallest positive time such that $F_{n-1}^{t_{n-1}^{(r)}}(\rho^0) \in S$ for $\rho^0 = (0, r) \in S$, $r = (r_1, \dots r_{n-1})$. Then

$$\begin{aligned} F_{n-1}^{t_{n-1}(r)}(\rho^0) &= (R_1(r), \dots, R_{n-2}(r), 0, r) \\ &= F_1^{R_1(r)} \circ \cdots \circ F_{n-2}^{R_{n-2}(r)}(\rho^0). \end{aligned}$$

Therefore $e_{n-1}(r) = R(r) = (R_1(r), \dots, R_{n-1}(r))$, $R_{n-1}(r) = t_{n-1}(r)$, is a stationary point of F^t, i.e. $F^{e_{n-1}(r)}(\rho) = \rho$ for any $\rho \in M_r$.

Moreover, $R_j(r)$ are smooth functions of $(p, q, \varphi_{n-1}, r_{n-1})$ and since they do not depend on φ_j, we obtain that R_j are smooth functions of r in $[0, \delta]^{n-2} \times [-\delta, \delta]$.

Now, for $(t, r) \in \mathbb{T}^{n-1} \times [0, \delta]^{n-2} \times [-\delta, \delta]$ we set

$$G_1(t, r) = F_{n-1}^{t_{n-1}R_{n-1}(r)}(t_1 + t_{n-1}R_1(r), \dots, t_{n-2} + t_{n-1}R_{n-2}(r), 0, r).$$

Thus we obtain a smooth map $\pi^{-1}G_1\pi$ in the coordinates (p, q, t_{n-1}, r_{n-1}):

$$\pi^{-1}G_1\pi(p, q, t_{n-1}, r_{n-1}) = F_{n-1}^{t_{n-1}R_{n-1}(r)}(M\binom{p}{q}, 0, r_{n-1})$$

where π is the respective polar change of the coordinates, M is the block-diagonal matrix $M = \operatorname{diag}(M_1, \dots, M_{n-2})$,

$$M_j = \begin{pmatrix} \cos(t_{n-1}R_j) & -\sin(t_{n-1}R_j) \\ \sin(t_{n-1}R_j) & \cos(t_{n-1}R_j) \end{pmatrix}, \qquad j = 1, \dots, n-2.$$

Since the vectors $e_j \in \mathbb{R}^{n-1}$, $j = 1, \dots, n-1$, are generators of the stationary group of $\mathbb{R}^{n-1} \ni t \to F^t(\rho)$, $\rho \in S$, we obtain that G_1 is a diffeomorphism of V_δ onto a neighbourhood U_δ of $\tilde{\mathscr{O}}$ in $T^*\partial\Omega$ for δ small enough. As in [3], we can construct some polar symplectic coordinates (φ, I) with respect to ω_1 in a neighbourhood of $\{r = 0\}$ starting from (t, r) which will give us the smooth exact symplectic transformation $G\colon V_\delta \to U_\delta$ in Proposition 4.2. Unlike [3], we work in a neighbourhood of $\{r = 0\}$ and not near a fixed torus and are interested in the smoothness of the respective maps near $\{r = 0\}$.

First we shall show that besides the equalities $\{r_j, r_k\}_1 = 0$, j, $k \le n-1$, we have

$$
\begin{aligned}
&\{r_j, t_k\}_1 = \delta_{jk}, \qquad j \le n-2, k \le n-1, \\
&\{r_{n-1}, t_j\}_1 = -R_j(r)/R_{n-1}(r), \qquad j \le n-2, \\
&\{r_{n-1}, t_{n-1}\}_1 = 1/R_{n-1}(r)
\end{aligned}
\tag{4.21}
$$

and

$$
\{t_j, t_k\}_1 = 0 \quad \text{for } j, k \le n-1. \tag{4.22}
$$

Equalities (4.21) follow from

$$
F_j^s G_1(t, r) = G_1(t + s e_j, r), \qquad j \le n-2,
$$

$$
F_{n-1}^s G_1(t, r) = G_1(t_1 - sR_1/R_{n-1}, \dots, t_{n-2} - sR_{n-2}/R_{n-1}, t_{n-1} + s/R_{n-1}, r).
$$

We shall prove equalities (4.22). From Jacobi's identity we have

$$
X^1_{r_i}\{t_j, t_k\}_1 = 0, \qquad j, k \le n-1, i \le n-2,
$$

$$
X^1_{r_{n-1}}\{t_j, t_k\}_1 = c_{jk}(r)
$$

where c_{jk} are smooth functions of $r \in [0, \delta)^{n-2} \times (-\delta, \delta)$. Hence

$$
\{t_j, t_k\}_1(F_{n-1}^{R_{n-1}(r)}(\rho)) = R_{n-1}(r)c_{jk}(r) + \{t_j, t_k\}_1(\rho), \qquad \rho \in S.
$$

On the other hand,

$$
\begin{aligned}
\{t_j, t_k\}_1(\rho) = \{t_j, t_k\}_1(F^{e_{n-1}}(\rho)) &= \{t_j, t_k\}_1(F_{n-1}^{R_{n-1}(r)}(\rho)) \\
&= R_{n-1}(r)c_{jk}(r) + \{t_j, t_k\}_1(\rho), \qquad \rho \in S.
\end{aligned}
$$

Thus $c_{jk}(r) = 0$, $j, k \le n-1$. Moreover, in U_0

$$
\begin{aligned}
\{t_j, t_{n-1}\}(\rho) &= \left\{ \varphi_j - \varphi_{n-1}\frac{R_j}{R_{n-1}}, \frac{\varphi_{n-1}}{R_{n-1}} \right\} \\
&= \varphi_{n-1}\left(\frac{\partial R_j}{\partial r_{n-1}} - \frac{\partial R_{n-1}}{\partial r_j} \right) \frac{1}{R^2_{n-1}}
\end{aligned}
$$

and since $X^1_{r_{n-1}} = \partial/\partial\varphi_{n-1}$ in U_0, we have $\{t_j, t_{n-1}\}(\rho) = 0$ for $\rho \in U_0$ and therefore for any $\rho \in U$. Analogously we obtain $\{t_j, t_k\}(\rho) = 0$, $\rho \in U$, j, $k \le n-2$, which proves (4.22).

We put $G_2(t,r) = (\varphi, I)$ where $\varphi_j = t_j$, $j \le n-1$, $I_j = r_j$, $j \le n-2$, $r_{n-1} = I_{n-1} + Q(I)$, $Q(0,t_0) = -t_0$. For the function $Q(I)$ we obtain the equations

$$\begin{aligned}
\{I_{n-1}, \varphi_j\}_1 &= -\frac{R_j(r)}{R_{n-1}(r)} - \{Q(I), t_j\}_1 \\
&= -\frac{R_j(r)}{R_{n-1}(r)} - \sum_{k=1}^{n-1} \frac{\partial Q}{\partial I_k}\{I_k, t_j\} \\
&= -\frac{R_j(r)}{R_{n-1}(r)} - \frac{\partial Q}{\partial I_j} - \frac{\partial Q}{\partial I_{n-1}}\{I_{n-1}, \varphi_j\}_1, \qquad j \le n-2,
\end{aligned}$$

$$\{I_{n-1}, \varphi_{n-1}\}_1 = \frac{1}{R_{n-1}(r)} - \frac{\partial Q}{\partial I_{n-1}}\{I_{n-1}, \varphi_{n-1}\}_1.$$

That is why we put $\partial Q(I)/\partial I_j = P_j(I)$, $j = 1, \dots, n-1$, where $P_j(I) = -R_j(r)/R_{n-1}(r)$ for $j \le n-2$ and $P_{n-1}(I) = 1/R_{n-1}(r) - 1$, $r = (I', I_{n-1} + Q(I))$.

These equations are solvable in a neighbourhood of $\{r = 0\}$ since P_j are smooth functions and $\partial P_j/\partial I_k = \partial P_k/\partial I_j$, j, $k = 1, \dots, n-1$ (cf. [3]). Indeed, we have

$$\frac{\partial P_j}{\partial I_k} = -\{t_k, \{r_{n-1}, t_j\}_1\}_1 = -\{t_j, \{r_{n-1}, t_k\}_1\}_1 = \frac{\partial P_k}{\partial I_j}.$$

Then the map $\pi^{-1} G_2 G_1 \pi$ is a diffeomorphism in V_δ.

Lemma 4.5. *There exists a diffeomorphism $G_3 \colon U_\delta \to U$ such that $G_3^* \omega_1 = \omega$ and $G_3 = \mathrm{Id} + G_4$ where $|G_4(\rho)| \le C|r'(\rho)|^{N+1/2}$.*

Proof. Following Moser [16], consider $\sigma_t = \sigma + t\beta$, $t \in [0,1]$ and $\omega_t = \omega - t\,d\beta$. From (4.15) it follows that ω_t is a nondegenerate 2-form in U if U is small enough. Let Y_t be a vector field determined by $\omega_t \lrcorner Y_t = \beta$ and $\Phi_t(\rho) = \exp(tY_t)(\rho)$. Here $\lrcorner$ denotes the inner product. Then from (4.15) we obtain

$$|\Phi_t(\rho) - \rho| \le C|r'(\rho)|^{N+1/2}, \qquad 0 \le t \le 1.$$

Moreover, we have

$$\frac{d}{dt}(\Phi_t)^* \omega_t = (\Phi_t)^*(-d\beta + d(\omega_t \lrcorner Y_t)) = 0,$$

hence $G_3 = \Phi_1$ is the diffeomorphism we seek.

Denote $G = G_3 \pi^{-1} G_2 G_1 \pi$. Then $G(V_\delta) = U$ is a neighbourhood of $\widetilde{\mathscr{O}}$ in $T^* \partial\Omega$ and

$$G^* \omega = \sum_{j=1}^{n-2} dp_j \wedge dq_j + dI_{n-1} \wedge d\varphi_{n-1}.$$

It remains to prove (4.6). Denote $\rho_j = F_j^{R_j}(\rho_{j-1})$, $1 \le j \le n$, where $\rho_0 \in S$ has polar coordinates $(0, r(I))$, $r(I) = (r', t_0\zeta) = (I', t_0\zeta(I))$. Then $\rho_n = \rho_0$

and $\gamma = \bigcup_{j=1}^{n-1} \gamma_j$, where $\gamma_j = \{F_j^t(\rho_{j-1}),\ 0 \le t \le R_{j-1}(r)\}$, is a cycle on M_r. Since the form $\sigma - id\varphi$ is closed in U_δ, we have

$$\int_\gamma \sigma - \int_\gamma I d\varphi = \int_\gamma \sigma - 2\pi I_{n-1} = c = \text{const}. \tag{4.23}$$

Therefore, for $(\varphi, I) \in \Pi^{n-1}$

$$2\pi I_{n-1} = \int_\gamma \sigma - C = \int_\gamma \sigma_1 + \int_\gamma \beta - C = \int_\gamma \sigma_1 + O(|I'|^{N+1/2}) - C. \tag{4.24}$$

On the other hand, $\rho_j = (R_1(r), \dots, R_j(r),\ 0, \dots, 0,\ r)$ and

$$\gamma_j = \{(R_1(r), \dots, R_{j-1}(r), tR_j(r), 0, \dots, 0, r); 0 \le t \le 1\}$$

for $j = 1, \dots, n-1$. Denote

$$\begin{aligned} L_j = \{(&R_1(sI', t_0, \zeta), \dots, R_{j-1}(sI', t_0\zeta), \\ &tR_j(sI', t_0\zeta), 0, \dots, 0, sI', t_0\zeta); 0 \le s \le 1, 0 \le t \le 1\}, \qquad j \le n-2, \\ L_{n-1} = \{&F_{n-1}^t(R_1(sI', t_0\zeta), \dots, R_{n-1}(sI', t_0\zeta), sI', t_0\zeta); \\ &\qquad\qquad 0 \le s \le 1, 0 \le t \le R_{n-1}(r)\} \end{aligned}$$

and

$$\gamma_\zeta = \{F_{n-1}^t(\underbrace{0, \dots, 0}_{2n-3 \text{ times}}, t_0\zeta); t \in \mathbb{R}^1\}.$$

Note that γ_ζ belongs to the cylinder of orbits of $X_{t_0\zeta}$. Now we have

$$\int_\gamma \sigma_1 = \int_{\gamma_\zeta} \sigma_1 - \sum_{j=1}^{n-1} \int_{L_j} \omega_1.$$

Moreover, L_{n-1} is isotropic so that $\int_{L_{n-1}} \omega_1 = 0$ while

$$\begin{aligned} \sum_{j=1}^{n-2} \int_{L_j} \omega_1 &= \sum_{j=1}^{n-2} \sum_{k=1}^{n-2} \int_{L_j} dI_k \wedge d\varphi_k = \sum_{j=1}^{n-2} \int_0^1 I_j R_j(sI', t_0\zeta)\, ds \\ &= \sum_{j=1}^{n-2} \int_0^1 I_j \frac{\partial A}{\partial I_j}(sI', \zeta)\, ds + O(|I'|^{N+1/2}) \\ &= A(I', \zeta) + O(|I'|^{N+1/2}) \end{aligned}$$

since $A(0, \zeta) = 0$. Finally, on the cylinder of orbits $L = \{F_{n-1}^t(0, \dots, 0, h); 0 \le h \le \zeta\}$ we have

$$\begin{aligned} \int_{\gamma_\zeta} \sigma_1 &= \int_{\gamma_0} \sigma_1 + \int_L \omega_1 = 2\pi t_0 + \int_L dh \wedge dt \\ &= 2\pi t_0 - \int_0^\zeta \int_0^{R_{n-1}(0,\zeta)} dt \wedge dh = 2\pi t_0 - \int_0^\zeta l(h)\, dh. \end{aligned}$$

Note that for $I' = 0$, $I_{n-1} = t_0$ (4.24) yields $C = 0$ in (4.23), thus G is exact symplectic and

$$2\pi I_{n-1} = 2\pi t_0 - \int_0^\zeta l(h)\,dh - A(I',\zeta) + O(|I'|^{N+1/2}).$$

Now if $\zeta_0(I)$ is a solution of (4.4)

$$2\pi I_{n-1} = 2\pi t_0 - \int_0^\zeta l^0(h)\,dh - A^0(I',\zeta), \zeta(0, t_0) = 0,$$

where $l^0(\zeta)$ and $A^0(I',\zeta)$ are Taylor's expansions of $l(\zeta)$ and $A(I',\zeta)$ with respect to ζ up to degrees M and $M+1$ respectively, M large enough, we obtain

$$|\zeta(\varphi, I) - \zeta_0(I)| \le C(|I'|^{N+1/2} + |\zeta_0|^{M+2})$$

which proves (4.6).

From the representation of the billiard ball map by means of the interpolating Hamiltonian ζ and from (4.6) we obtain

$$B(\varphi, I) = \exp(-\sqrt{\zeta} X_{\zeta_0})((\varphi, I) + R(\varphi, I)) \quad \text{in } V_\delta \cap \{\zeta \ge 0\}$$

where

$$R(\varphi, I) = (O(|I'|^{N-1/2}) + O(\zeta_0^{M+2}), O(|I'|^{N+1/2}) + O(\zeta_0^{M+2}))$$

and $R(\varphi, I) = R_1(\varphi, \sqrt{I'}, \sqrt{\zeta})$ where R_1 is a smooth function of its arguments for $(\varphi, I) \in \mathbb{T}^{n-1} \times \mathbb{D}^{n-1}$. Here we have denoted

$$\mathbb{D}^{n-1} = \{I \in \mathbb{R}^{n-1}; 0 \le I_j < \delta, j = 1, \dots, n-2, |I_{n-1} - t_0| < \delta,$$
$$\zeta_0(I) > 0, |I'| \le C\zeta_0(I)\},$$

where $C > 1$ is such that $|\zeta(\varphi, I) - \zeta_0(I)| \le \frac{1}{2}(\zeta_0(I))^{M+2}$ for $I \in \mathbb{D}^{n-1}$ and $\varphi \in \mathbb{T}^{n-1}$ (here we have used (4.5)).

Then we have

$$B(\varphi, I) = \exp(-\sqrt{\zeta_0} X_{\zeta_0})((\varphi, I) + g(\varphi, I)) \tag{4.25}$$

for $(\varphi, I) \in \mathbb{T}^{n-1} \times \mathbb{D}^{n-1}$ where

$$g(\varphi, I) = (O(|I'|^{N-1/2}), O(|I'|^{N+1/2})) + O(\zeta_0^{M+2})$$

and g is symplectic in $\mathbb{T}^{n-1} \times \mathbb{D}^{n-1}$.

4.3. Denote

$$\mathbb{D}_a^{n-1} = \{I \in \mathbb{R}^{n-1}, C_1 a \le I_j \le C_2 a, j = 1, \dots, n-2,$$
$$C_3 a^{2b} \le \zeta_0(I) \le C_4 a^{2b}\} \tag{4.26}$$

where $0 < C_1 < 1 < C_2$, $0 < C_3 < 1 < C_4$, $0 < b < 1/2$ small enough. Henceforth C_j will be constants which do not depend on a but they will depend on the concrete circumstances. Note that for $a \le a_0$, a_0 small enough, we have $\mathbb{D}_a^{n-1} \subset \mathbb{D}^{n-1}$. We shall denote $\mathbb{A}_a^{n-1} = \mathbb{T}^{n-1} \times \mathbb{D}_a^{n-1}$.

Denote for $\Omega \subset \mathbb{R}^m$, $\rho > 0$,

$$\Omega + \rho = \{z \in \mathbb{C}^m ;\ \mathrm{dist}(z, \Omega) \le \rho\} \tag{4.27}$$

and set $H^0(I) = -\frac{2}{3}(\zeta_0(I))^{3/2}$, $H(\varphi, I) = -\frac{2}{3}(\zeta(\varphi, I))^{3/2}$.

Proposition 4.6. *For a_0 small enough we have*

$$\|H^0_{II}(I)\|, \|(H^0_{II}(I))^{-1}\| \le C_1' a^{-b}, \qquad I \in \mathbb{D}_a^{n-1} + \rho, \tag{4.28}$$

$$|\det H^0_{II}(I)| \ge C_2' a^{b(n-3)} \tag{4.29}$$

for each $a \in (0, a_0)$, $0 < \rho \le a/2$, where $\|\cdot\|$ is the norm of the respective $(n-1) \times (n-1)$-matrix. Moreover, the map

$$\mathbb{D}_a^{n-1} + \rho \ni I \to H_I^0(I)$$

is a diffeomorphism.

Proof. We have $H_I^0(I) = -\zeta_0^{1/2} \operatorname{grad} \zeta_0(I)$. But

$$\operatorname{grad} \zeta_0(I) = -(l^0(\zeta_0) + A^0_\zeta(I', \zeta_0))^{-1}(A^0_{I'}(I', \zeta_0), -1),$$

thus

$$H_I^0 = \zeta_0^{1/2}(l^0(\zeta_0) + A^0_\zeta(I', \zeta_0))^{-1}(A^0_{I'}(I', \zeta_0), -1).$$

Set $K = \zeta_0(l^0(\zeta_0) + A^0_\zeta(I', \zeta_0))^{-2}$. We have $\partial K(0)/\partial \zeta_0 \ne 0$ and this allows us to regard the map $I \to H_I^0(I)$ as a composition of the following three maps:

$$(I', I_{n-1}) \to (I', \zeta_0) \to (I', K) \to H_I^0(I).$$

This gives us the representation

$$H^0_{II}(I) = \frac{\partial H_I^0}{\partial(I', K)} \frac{\partial(I', K)}{\partial(I', \zeta_0)} \frac{\partial(I', \zeta_0)}{\partial(I', I_{n-1})}.$$

The first two maps are diffeomorphisms in a sufficiently small neighbourhood of $\mathcal{O}$ and for the respective functional matrices we have

$$\frac{\partial(I', K)}{\partial(I', \zeta_0)} = \begin{pmatrix} E_{n-2} & O_{n-2} \\ K_{I'} & \partial K/\partial \zeta_0 \end{pmatrix}$$

$$\frac{\partial(I', \zeta_0)}{\partial(I', I_{n-1})} = \begin{pmatrix} E_{n-2} & O_{n-2} \\ \operatorname{grad}_{I'} \zeta_0 & \partial \zeta_0/\partial I_{n-1} \end{pmatrix}$$

where E_{n-2} is the unit $(n-2)$-matrix and O_{n-2} is the zero $(n-2)$-dimensional column matrix. Thus it remains to consider the matrix $\partial H_I^0/\partial(I', K)$ for $I \in \mathbb{D}_a^{n-1} + \rho$, i.e. when the variables (I', K) run over a set of the form

$$[\widetilde{C}_1 a, \widetilde{C}_2 a]^{n-2} \times [\widetilde{C}_3 a^{2b}, \widetilde{C}_4 a^{2b}] + \widetilde{C}_5 \rho.$$

We have $H_I^0 = K^{1/2}(A_{I'}^0(I',\zeta_0), -1)$ where $\zeta_0 = KA^1(I',K)$ and the function $A^1(I',K)$ satisfies $A^1(0,0) \neq 0$. Then for the entries of the matrix $\partial H_I^0/\partial(I',K)$ we have

$$A_{jk} = K^{1/2}\left(\frac{\partial^2 A^0(I',\zeta_0)}{\partial I_j \partial I_k} + K\frac{\partial A^0(I',\zeta_0)}{\partial I_j}\cdot\frac{\partial A^1(I',K)}{\partial I_k}\right)$$

for $1 \le j,\ k \le n-2$,

$$A_{j,n-1} = \frac{1}{2}K^{-1/2}\frac{\partial A^0(I',\zeta_0)}{\partial I_j} + K^{1/2}\frac{\partial^2 A^0(I',\zeta_0)}{\partial I_j \partial \zeta}\left(A^1(I',K) + K\frac{\partial A^1(I',K)}{\partial K}\right)$$
$$\text{for } j = 1, \dots, n-2,$$

$A_{n-1,k} = 0$ for $k = 1, \dots, n-2$, $A_{n-1,n-1} = -\frac{1}{2}K^{-1/2}$.

Now it is easy to see that the validity of Proposition 4.6 follows from the invertibility of the matrix $A_{I'I'}^0(I',\zeta)$ for I', ζ small enough which is a consequence of the nondegeneracy assumption (2.3). This completes the proof of Proposition 4.6.

5. Main theorem

Denote $\Omega_a = \{\omega = H_I^0(I);\ I \in \mathbb{D}_a^{n-1}\}$ and let the numbers σ and γ be such that $\sigma > n-1$ and $\gamma > 0$.

Then denote

(5.1) $\Omega_a^\gamma = \{\omega \in \Omega_a; |\langle\omega, k'\rangle - k_n| \ge \gamma|k|^{-\sigma}$ for any $k = (k', k_n) \in \mathbb{Z}^n \setminus \{0\}\}$.

The next lemma shows that the family of invariant tori we are going to obtain has a positive Lebesgue measure.

Lemma 5.1. *Let $\gamma < \widetilde{C}_6 a^{1+lb}$, $\widetilde{C}_6 > 0$, $l > 0$. Then for a small enough we have*

$$\operatorname{mes}\Omega_a^\gamma \ge (1 - \tilde{c}_1 a^{lb})\tilde{c}_2 a^{n-2+b}.$$

Proof. By analytic-geometrical arguments we obtain $\operatorname{mes}\Omega_a \ge \widetilde{C}_7 a^{n-2+b}$. Then by arguments close to those in [1] we find $\operatorname{mes}(\Omega_a \setminus \Omega_a^\gamma) \le \widetilde{C}_8 \gamma a^{n-3+b}$ where $\widetilde{C}_8$ depends only on n and σ. This yields the desired inequality.

Denote by $\sigma_\gamma(\varphi, I)$ the map $(\varphi, I) \to (\varphi, \gamma I)$ and by $\|\cdot\|_{s,\mathbb{T}^{n-1}\times\Omega}$, $s > 0$, the respective Hölder norms of the functions in $\mathbb{T}^{n-1} \times \Omega$ as well as

$$\|f\|_{s,\mathbb{T}^{n-1}\times\Omega;\gamma} = \|f \circ \sigma_\gamma\|_{s,\sigma_\gamma^{-1}(\mathbb{T}^{n-1}\times\Omega)}$$

(see [19]).

Let $C_5 a^{1+lb} < \gamma_0 < C_6 a^{1+lb}$, $1 < C_5 < C_6$, $l > 0$, and denote $\gamma = \gamma_0 a^{3b}$.

Theorem 1 follows from the following theorem.

Theorem 5.2. *Let $N \ge 2$, $0 < a < a_0$ where a_0 is small enough. Then there exists an exact symplectic diffeomorphism $U\colon \mathbb{A}_a^{n-1} \to \mathbb{A}_a^{n-1}$ and a function K,*

$K(I) > 0$ in $\mathbb{D}_a^{n-1}$, of class C^∞ such that

$$(5.2)\qquad U^{-1}BU(\varphi, I) = (\varphi + \tau_I(I), I), \qquad \tau(I) = -\tfrac{2}{3}K(I)^{3/2},$$

for any $\varphi \in \mathbb{T}^{n-1}$, $I \in \mathbb{D}_{a,\gamma_0}^{n-1} = \{I \in \mathbb{D}_a^{n-1}; \tau_I(I) \in \Omega_a^{\gamma_0}\}$. Moreover,

$$(5.3)\qquad \|K - \zeta_0\|_{p, \mathbb{D}_a^{n-1};\gamma} \le C_p a^{N-1/2-(l+2)b}$$

and the generating function $S(\theta, I)$ of U satisfies

$$(5.4)\qquad \|S\|_{p, \mathbb{A}_a^{n-1};\gamma} \le C_p a^{N-1/2-(l+1)b} \quad \textit{for any } p \ge 0.$$

In order to prove Theorem 5.2 we reduce the problem to finding invariant tori for the flow of a suitable Hamiltonian. For this purpose we use some arguments from [4, 5].

Now we shall use the fact that the map $\mathrm{Id} + g$ from (4.25) is exact symplectic. This is a consequence of (4.25) and the following assertion proved in [6]:

Proposition 5.3. *The billiard ball map B is exact symplectic in V_δ.*

Let $\eta \in C^\infty(\mathbb{R}^1)$, $\eta = 0$ in a neighbourhood of 0, $\eta = 1$ in a neighbourhood of 2π and let φ_t be exact symplectic with a generating function $\eta(t)S$ where S generates $\varphi_{2\pi} = \mathrm{Id} + g$. Denote by B_t the exact symplectic map $B_t = \exp(tX_{H^0}) \circ \varphi_t$, $t \in \mathbb{R}^1$. Let $\xi_t = dB_t/dt \circ (B_t)^{-1}$ be the respective vector field which is well defined for $t \in \mathbb{T}^1$ since $\xi_t = X_{H^0}$ in some neighbourhoods of $t = 0$ and $t = 2\pi$. Following Douady [4], we obtain $\omega_{n-1} \lrcorner\, \xi_t = dh_t$ for a suitable C^∞ function h_t, $t \in \mathbb{T}^1$. Moreover, using (4.25) we have

$$(5.5)\ h_t(\varphi, I) = H^0(I) + \sqrt{\zeta_0} \sum_{|\alpha| = 2N+1} (I')^{\alpha/2} Q_\alpha\left(t, \varphi, \sqrt{I_1}, \dots, \sqrt{I_{n-2}}, \sqrt{\zeta_0}\right) + \zeta_0^{M+5/2} Q'\left(t, \varphi, \sqrt{I_1}, \dots, \sqrt{I_{n-2}}, \sqrt{\zeta_0}\right)$$

for $(\varphi, I) \in \mathbb{A}_a^{n-1}$ where Q_α, Q' are smooth functions of $(t, \varphi, \sqrt{I_1}, \dots, \sqrt{I_{n-2}}, \sqrt{\zeta_0})$.

Denote $y' = \varphi$, $y = (y', y_n)$, $\eta' = I$, $\eta = (\eta', \eta_n)$ where the variable η_n runs over a neighbourhood of 0 in $\mathbb{R}^1$ and $y_n \in \mathbb{T}^1$. Set

$$(5.6)\qquad \widetilde{H}^0(\eta) = H^0(\eta') + \eta_n, \qquad \widetilde{H}(y, \eta) = h_{y_n}(y', \eta') + \eta_n.$$

Denote by $(F^t)_{t \in \mathbb{R}^1}$ the flow of $X_{\widetilde{H}}$ and let $\mathbb{A}' = \{(y, \eta) \in \mathbb{A}_a^n;\ \widetilde{H}(y, \eta) = 0,\ y_n = 0\}$ where $\mathbb{A}_a^n = \mathbb{T}^n \times \mathbb{D}_a^n$,

$$\mathbb{D}_a^n = \{\eta \in \mathbb{R}^n; C_7 a \le \eta_j \le C_8 a,\ j = 1, \dots, n-2,\ C_9 a^{2b} \le \zeta_0(\eta') \le C_{10} a^{2b}, |\eta_n| \le C_{11}\}$$

and C_j are some positive constants such that $C_7 < 1 < C_8$, $C_9 < 1 < C_{10}$. It is easy to see that $\mathbb{A}'$ has the form

$$\mathbb{A}' = \{(y', 0, \eta', -H^0(\eta')); (y', \eta') \in \mathbb{A}_a^{n-1}\}.$$

Denote by $\imath\colon \mathbb{A}_a^{n-1} \to \mathbb{A}$ the map

$$\imath(y', \eta') = (y', 0, \eta', -h_0(y', \eta')) = (y', 0, \eta', -H^0(\eta')).$$

It is easy to see that $F^{2\pi}\colon \mathbb{A}' \to \mathbb{A}'$ satisfies

$$\imath^{-1} \circ F^{2\pi} \circ \imath = B. \tag{5.7}$$

Denote $H_0' = \widetilde{H}^0 + (\widetilde{H}^0)^2$, $H' = \widetilde{H} + \widetilde{H}^2$.

Now, from Proposition 4.6 we obtain

$$\left\| \frac{\partial^2 H_0'}{\partial \eta^2} \right\|, \left\| \left(\frac{\partial^2 H_0'}{\partial \eta^2} \right)^{-1} \right\| \le C a^{-b} \quad \text{in } \mathbb{D}_a^n + \rho \tag{5.8}$$

where $0 < a \le a_0$, $b > 0$ small enough are fixed and $\rho \le a/2$. Moreover, in $\mathbb{A}_a^n$ we have

$$\begin{aligned} H'(y, \eta) - H_0'(\eta) &= (h_{y_n}(y', \eta') - H^0(\eta'))(1 + \widetilde{H}(y, \eta) + H^0(\eta')) \\ &= a^b O(|\eta''|^{N+1/2}) \end{aligned} \tag{5.9}$$

where $\eta'' = (\eta_1 \dots, \eta_{n-2})$ which follows from (5.5) for M large enough.

Introduce the sets $\Omega' = \{\omega = \operatorname{grad} H_0'(\eta); \eta \in \mathbb{D}_a^n\}$ and $\Omega'_{\gamma_1} = \{\omega \in \Omega';$ $|\langle \omega, k \rangle| \ge \gamma_1 |k|^{-\sigma}$ for any $k \in \mathbb{Z}^n \setminus \{0\}\}$ where σ is as above and $\gamma_1 = C\gamma a^{-3b} = C\gamma_0$, $0 < C < 1$.

We reduce the proof of Theorem 5.2 to

Theorem 5.4. *Let $H_0'(\eta)$ be an analytic function in $\mathbb{D}_a^n + \rho$ such that*

$$\left\| \frac{\partial^2 H_0'}{\partial \eta^2} \right\|_{\mathbb{D}_a^n + \rho}, \left\| \left(\frac{\partial^2 H_0'}{\partial \eta^2} \right)^{-1} \right\|_{\mathbb{D}_a^n + \rho} \le R, \qquad R \ge 1, \tag{5.10}$$

and let $\partial H_0'/\partial \eta\colon \mathbb{D}_a^n + \rho \to \mathbb{C}^n$ be invertible on $\mathbb{D}_a^n + \rho$.

Then for any fixed $\lambda > \sigma + 1 > n$, $\alpha > 1$, $\alpha \notin \Lambda = \{i/\lambda + j;\ i, j \ge 0$ integer$\}$ there exists $\delta \le C_1 R^{-3}$ such that if $H' \in C^\infty(\mathbb{A}_a^n)$ and

$$R \|H' - H_0'\|_{s, \mathbb{A}_a^n; \gamma_1} \le \gamma_1^2 \delta, \qquad s = \alpha\lambda + \lambda + \sigma,\ 0 < \gamma_1 \le \rho/R, \tag{5.11}$$

then

(i) *there exists a C^∞ function $\widetilde{S}(Y, \eta)$ and a nondegenerate Hamiltonian $K'(\eta)$ such that*

$$H'(Y, \eta - \widetilde{S}_Y(Y, \eta))|_{\mathbb{A}_{a,\gamma_1}^n} = K'(\eta)$$

where $\mathbb{A}_{a,\gamma_1}^n = \mathbb{T}^n \times \mathbb{D}_{a,\gamma_1}^n$ and $\mathbb{D}_{a,\gamma_1}^n = \{\eta \in \mathbb{R}^n; K_\eta'(\eta) \in \Omega'_{\gamma_1}\}$;

(ii) $\|\widetilde{S}\|_{\tilde{\beta}, \mathbb{A}_a^n; \gamma_1} \le C_\beta \gamma_1^{-1} R^{\beta+1} \|H' - H_0'\|_{p, \mathbb{A}_a^n; \gamma_1}$ *where* $p = \beta\lambda + \lambda + \sigma$, $\tilde{\beta} = \beta - (\lambda - \sigma)/\lambda \notin \Lambda$, $\beta \ge \alpha$;

(iii) *$\widetilde{S}$ generates a canonic map $\widetilde{T}$ on $\mathbb{A}_a^n$ for $|\widetilde{S}_{Y\eta}|$ small enough.*

In fact, Theorem 5.4 is proved in [19, Theorem A]. We just follow the dependence of the various constants in [19] on R and ρ as well as the exponent of γ_1 in (ii) (for more details see the Appendix).

Proof of Theorem 5.2. First we estimate $H' - H_0'$. Using (5.9) and the fact that $H' - H_0'$ is a smooth function of $(y, \sqrt{\eta_1}, \dots, \sqrt{\eta_{n-2}}, \sqrt{\zeta_0}, \eta_n)$ in $\mathbb{A}_a^n$, we obtain

$$(5.12) \qquad R\|H' - H_0'\|_{p, \mathbb{A}_a^n; \gamma_1} \le Ca^{N+1/2} \le \gamma_1^2 \delta \quad \text{for any } p > 0$$

where $\delta = a^{4b}$, $R = Ca^{-b}$, $N \ge 2$ and $0 < b < 1/(4l+8)$. Thus Theorem 5.4 holds for H' and H_0' and from (ii) we obtain

$$(5.13) \qquad \|\widetilde{S}\|_{p, \mathbb{A}_a^n; \gamma_1} \le C_p a^{N-1/2-(p+l+1)b} \quad \text{for any } p > 0.$$

Note that $\widetilde{S}_{Y\eta}$ is small enough if a_0 is small enough and $N \ge 2$ since in view of (5.13) we have $|\widetilde{S}_{Y\eta}| \le Ca^{N-3/2-(2+2l)b}$ in $\mathbb{A}_a^n$. Then $\widetilde{S}$ generates a canonic map $\widetilde{T}$, i.e.

$$\operatorname{graph} \widetilde{T} = \{(Y - \widetilde{S}_\eta(Y, \eta), \eta; Y, \eta - \widetilde{S}_Y(Y, \eta))\}.$$

Now we shall concentrate our efforts on the construction of the function K and the symplectic map U of Theorem 5.2 starting from the nongenerate Hamiltonian K' and the symplectic transformation $\widetilde{T}$.

First we shall need some preliminaries.

Let us denote $K_1(y, \eta) = \widetilde{H}(\widetilde{T}(y, \eta))$. Then $K_1 + K_1^2 = K'$ and K_1 does not depend on y for any $\eta \in \mathbb{D}_{a,\gamma}^n$. Set $K_2(\eta) = K_1(0, \eta)$.

The manifold $\Sigma_0 = \{(y, \eta) \in \mathbb{A}_a^n;\ \widetilde{H}(y, \eta) = 0\}$ is contained in $\{(y, \eta) \in \mathbb{A}_a^n;\ H'(y, \eta) = 0\}$. Moreover, if $\Lambda_\eta \cap \Sigma_0 \neq \emptyset$ for some invariant torus $\Lambda_\eta = \widetilde{T}(\mathbb{T}^n \times \{\eta\})$, then $\Lambda_\eta \subset \Sigma_0$ and

$$F^t|_{\Lambda_\eta} = \exp(tX_{H'})|_{\Lambda_\eta} \quad \text{where } F^t(\rho) = \exp(tX_{\widetilde{H}})(\rho) = (y^t(\rho), \eta^t(\rho)).$$

Denote $\widetilde{\Sigma} = \widetilde{T}^{-1}(\Sigma_0) = \{(y, \eta) \in \mathbb{A}_a^n; K_1(y, \eta) = 0\}$.

Since $\widetilde{T}$ is a canonic map, we have

$$(5.14) \qquad F^t(\widetilde{T}(y, \eta)) = \widetilde{T}(y + t \operatorname{grad} K_2(\eta), \eta)$$

for $(y, \eta) \in \mathbb{A}_{a,\gamma_1}^n \cap \widetilde{\Sigma}$, i.e. when $y \in \mathbb{T}^n$, $\eta \in \mathbb{D}_{a,\gamma_1}^n$ and $K_2(\eta) = 0$. We write down $\widetilde{T}(y, \eta) = (p(y, \eta), q(y, \eta))$. From (5.6) we have $\dot{y}_n = \partial\widetilde{H}/\partial\eta_n = 1$, thus $y_n^t = y_n^0 + t$ and we have

$$(5.15) \qquad p_n(y, \eta) + t = p_n(y + t \operatorname{grad} K_2(\eta), \eta) \quad \text{for any } t \in \mathbb{R}^1.$$

As in [4] we conclude that $\operatorname{grad}_y p_n(y, \eta)$ does not depend on $y \in \mathbb{T}^n$ if $\eta \in \mathbb{D}_{a,\gamma_1}^n$ and $K_2(\eta) = 0$. Therefore, $p_n(y, \eta) = y_n + f(\eta)$ for some function f on $\mathbb{D}_{a,\gamma_1}^n$. On the other hand,

$$\widetilde{T}(-\widetilde{S}_\eta(0, \eta), \eta) = (0, \eta - \widetilde{S}_Y(0, \eta)), \qquad (0, \eta) \in \mathbb{A}_{a,\gamma_1}^n \cap \widetilde{\Sigma},$$

thus

$$-\widetilde{S}_{\eta_n}(0,\eta)+f(\eta)=p_n(-\widetilde{S}_\eta(0,\eta),\eta)=0$$

and $f(\eta)=\widetilde{S}_{\eta_n}(0,\eta)$. From (5.15) we obtain

$$y_n+f(\eta)+t=y_n+t\frac{\partial K_2}{\partial \eta_n}(\eta)+f(\eta),$$

thus

$$\frac{\partial K_2}{\partial \eta_n}(\eta)=1,\qquad \eta\in\mathbb{D}^n_{a,\gamma_1} \tag{5.16}$$

Denote $g(\eta)=\widetilde{S}_\eta(0,\eta)$, $\eta\in\mathbb{D}^n_a$; then the map $T_1(y,\eta)=(y-g(\eta),\eta)$ is symplectic with a generating function $\widetilde{S}(0,\eta)$. Moreover, the map $\widetilde{T}\circ T_1$ is generated by the function $\widetilde{S}(Y,\eta)-\widetilde{S}(0,\eta)$ which satisfies (ii) and $\widetilde{H}(\widetilde{T}(T_1(y,\eta)))=K_2(\eta)$ for $\eta\in\mathbb{D}^n_{a,\gamma_1}$.

Thus we can suppose that

$$p_n(y,\eta)=y_n \tag{5.17}$$

for $(y,\eta)\in\mathbb{A}^n_{a,\gamma_1}\cap\widetilde{\Sigma}$ and y_n small enough.

Then Theorem 5.4(i), (5.14) and (5.17) yield $p_n(y',0,\eta)=0$ and

$$F^{2\pi}(\widetilde{T}(y',0,\eta))=\widetilde{T}(y'+\operatorname{grad}_{\eta'}K_2(\eta),2\pi,\eta) \tag{5.18}$$

for $(y',0,\eta)\in\widetilde{\Sigma}\cap\mathbb{A}^n_{a,\gamma_1}$.

We turn now to the construction of the function $K(I)$ in Theorem 5.2. First we prove that

$$\|K_2-\widetilde{H}^0\|_{p,\mathbb{D}^n_a;\gamma_2}\le C_p a^{N-1/2-(l+1)b}\quad\text{for any } p\ge 0,\ \gamma_2=\gamma_1 a^b. \tag{5.19}$$

Indeed, we have

$$\begin{aligned}K_2(\eta)-\widetilde{H}^0(\eta)&=(\widetilde{H}(0,\eta-\widetilde{S}_Y(0,\eta))-\widetilde{H}^0(\eta-\widetilde{S}_Y(0,\eta)))\\&\quad+(\widetilde{H}^0(\eta-\widetilde{S}_Y(0,\eta))-\widetilde{H}^0(\eta)).\end{aligned}$$

We evaluate the first addend by (5.12) and (5.13) using the following estimate for the norm of a composite function $F\circ G$:

$$\|F\circ G\|_{p,\gamma_1}\le\|F\|_{p,\gamma_1}\sum_{\alpha=0}^{p-1}c_{p,\alpha}\|DG\|^{p-\alpha}_{\alpha,\gamma_1},\qquad p\ge 1, \tag{5.20}$$

where DG is the matrix of the first derivatives of G and $C_{p,p-1}=1$. For the sake of simplicity we have dropped out the dependence of the above norms on the domain $\mathbb{D}^n_a$.

For the second term we have

$$\widetilde{H}^0(\eta-\widetilde{S}_Y(0,\eta))-\widetilde{H}^0(\eta)=-\left\langle \widetilde{S}_Y(0,\eta),\int_0^1(\operatorname{grad}\widetilde{H}^0)(\eta-t\widetilde{S}_Y(0,\eta))\,dt\right\rangle$$

whose (p, γ_1)-norm can be estimated by $C_p a^\beta$, $\beta = N - 1/2 - (p + l + 1)b$, using again (5.13) and (5.20). This proves (5.19).

Now, for $N \geq 2$, we can use the implicit function theorem to solve the equation $K_2(\eta) = 0$ with respect to η_n. We find a smooth function $\tau(\eta')$ choosing the constants C_j in the definition of $\mathbb{D}_a^{n-1}$ and $\mathbb{D}_a^n$ appropriately so that the set $\Sigma_1 = \{\eta; K_2(\eta) = 0\}$ can be represented as $\Sigma_1 = \{(\eta', -\tau(\eta'));\ \eta' \in \mathbb{D}_a^{n-1}\}$. Moreover, making use of (5.19) and (5.20), we find

$$(5.21) \qquad \|\tau + \tfrac{2}{3}\zeta_0^{3/2}\|_{p, \mathbb{D}_a^{n-1}; \gamma_2} \leq C_p a^{N-1/2-(l+1)b}, \qquad p \geq 0.$$

Thus $\tau < 0$ in $\mathbb{D}_a^{n-1}$ and the function $K(\eta') = (-\frac{3}{2}\tau(\eta'))^{2/3}$ is smooth and positive in $\mathbb{D}_a^{n-1}$. We shall prove that K satisfies the requirements of Theorem 5.2. First, in view of (5.21) and the equality

$$K - \zeta_0 = \zeta_0((1 - \tfrac{3}{2}\zeta_0^{-3/2}R)^{2/3} - 1), \qquad R = \tau + \tfrac{2}{3}\zeta_0^{3/2},$$

we obtain

$$(5.22) \qquad \|K - \zeta_0\|_{p, \mathbb{D}_a^{n-1}; \gamma_2} \leq C_p a^{N-1/2-(l+2+2p)b}$$

which yields (5.3) for $\gamma = \gamma_2 a^{2b} = \gamma_1 a^{3b}$.

It remains to construct the symplectic diffeomorphism U of Theorem 5.2. We shall write down explicitly the generating function of U starting from $\widetilde{S}$.

Lemma 5.5. *Suppose that* $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1} = \{\eta' \in \mathbb{D}_a^{n-1}; \operatorname{grad}\tau(\eta') \in \Omega_{\gamma_0}\}$. *Then* $(\eta', -\tau(\eta')) \in \mathbb{D}_{a,\gamma_1}^n$ *and* $\operatorname{grad} K_2(\eta', -\tau(\eta')) = (\operatorname{grad}\tau(\eta'), 1)$ *for* a_0 *small enough.*

Proof. We have

$$(\operatorname{grad} K_2)(\eta', -\tau(\eta')) = (\operatorname{grad}\tau(\eta'), 1)(\partial K_2/\partial\eta_n)(\eta', -\tau(\eta'))$$

which yields $(\operatorname{grad} K_2)(\eta', -\tau(\eta')) \in \Omega'_{\gamma_1}$ since $\gamma_1 < \gamma_0$ and

$$|\partial K_2/\partial\eta_n - 1| \leq C a^{N-3/2-(2l+3)b}.$$

Therefore

$$(\operatorname{grad} K')(\eta', -\tau(\eta')) = (\operatorname{grad} K_2)(\eta', -\tau(\eta')) \in \Omega'_{\gamma_1}$$

and $(\partial K_2/\partial\eta_n)(\eta', -\tau(\eta')) = 1$ in view of (5.16) which proves the claim.

Now using (5.17), we obtain $\widetilde{T}(y', 0, \eta', -\tau(\eta')) \in \mathbb{A}'$ for any $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$, $y' \in \mathbb{T}^{n-1}$ where $\mathbb{A}' = \Sigma_0 \cap \{y_n = 0\}$ has the form

$$\mathbb{A}' = \{(y', 0, \eta', -H^0(\eta')); (y', \eta') \in \mathbb{A}_a^{n-1}\}.$$

Moreover, $F^{2\pi} = \imath B \imath^{-1}: \mathbb{A}' \to \mathbb{A}'$ and from (5.18) and Lemma 5.5 we have

$$(5.23) \qquad \imath B \imath^{-1}(\widetilde{T}(y', 0, \eta', -\tau(\eta'))) = \widetilde{T}(y' + \operatorname{grad}\tau(\eta'), 2\pi, \eta', -\tau(\eta'))$$

for any $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$ and $y \in \mathbb{T}^{n-1}$.

Denote by $\imath_1$ the immersion

$$\imath_1(y', \eta') = (y', 0, \eta', -\tau(\eta')) \in \mathbb{T}^n \times \Sigma_1, \qquad (y', \eta') \in \mathbb{A}_a^{n-1},$$

and set $S(Y', \eta') = \widetilde{S}(Y', 0, \eta', -\tau(\eta'))$. This function satisfies (5.4) in view of (5.13) and (5.21). Denote by U the symplectic map generated by S. Then U maps $\mathbb{A}_a^{n-1}$ into a set of the same type. Moreover, $U(y', \eta') = \imath^{-1} \circ \widetilde{T} \circ \imath_1(y', \eta')$ for $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$ since $(\partial \widetilde{S}/\partial \eta_n)(Y, \eta', -\tau(\eta')) = 0$ for $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$ in view of (5.17) and Lemma 5.5. Now (5.2) follows directly from (5.23). This completes the proof of Theorem 5.2.

Remark 5.1. Instead of $K_2(\eta)$ we can use any function $\widetilde{K}_2(\eta) = K_2(\eta) + g(\eta)$ such that $g(\eta) = 0$ on $\mathbb{D}_{a,\gamma_1}^n$ and

$$\|g\|_{p,\gamma} \le C_p a^{N-1/2-(l+1)b} \quad \text{in } \mathbb{D}_a^n. \tag{5.24}$$

Then the corresponding solution $\tilde{\tau}(\eta')$ of the equation $\widetilde{K}_2(\eta', \eta_n) = 0$ with respect to η_n equals $\tau(\eta')$ for any $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$ and the sets $\mathbb{D}_{a,\gamma_0}^{n-1}$ and $\widetilde{\mathbb{D}}_{a,\gamma_0}^{n-1}$ defined by τ and $\tilde{\tau}$ coincide.

Indeed, writing down $\tilde{\tau}(\eta') = \tau(\eta') + r(\eta')$, $K_2(\eta) = (\eta_n + \tau(\eta'))h(\eta)$ with some function $h(\eta) \neq 0$ in a neighbourhood of $\mathbb{D}_a^n$ and

$$\widetilde{K}_2(\eta) = (\eta_n + \tau(\eta') + g_1(\eta))h(\eta), \qquad g_1(\eta) = g(\eta)/h(\eta),$$

we have

$$r(\eta') = g_2(\eta', r(\eta')), \qquad g_2(\eta) = g_1(\eta', -\tau(\eta') - \eta_n).$$

The function g_2 satisfies the estimate

$$\|g_2\|_{p,\gamma_2} \le C_p a^{N-1/2-(l+1)b} \tag{5.25}$$

in view of (5.21) and (5.24). Then $(\eta', r(\eta')) \in \mathbb{D}_a^n$ for $\eta' \in \mathbb{D}_a^{n-1}$ and by Lemma 5.5 we obtain the equality

$$r(\eta') = g_2(\eta', r(\eta')) - g_2(\eta', 0) = r(\eta')v(\eta') \tag{5.26}$$

for any $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$. Here

$$v(\eta') = \int_0^1 \frac{\partial g_2}{\partial \eta_n}(\eta', sr(\eta'))\, ds = O(a^{N-3/2-(2l+3)b}),$$

thus $|v(\eta')| < 1$ for $0 < a \le a_0$ and a_0 small enough. Now (5.26) implies $r(\eta') = 0$ on $\mathbb{D}_{a,\gamma_0}^{n-1}$. Since $\mathbb{D}_{a,\gamma_0}^{n-1}$ has no isolated points, we have $\operatorname{grad} \tilde{\tau}(\eta') = \operatorname{grad} \tau(\eta')$ for $\eta' \in \mathbb{D}_{a,\gamma_0}^{n-1}$. On the other hand, the map $\mathbb{D}_a^{n-1} \ni \eta' \to \operatorname{grad} \tilde{\tau}(\eta')$ is injective in view of Proposition 4.6 and (5.21) which proves the relation $\mathbb{D}_{a,\gamma_0}^{n-1} = \widetilde{\mathbb{D}}_{a,\gamma_0}^{n-1}$.

Proof of Theorem 1. Set $a_j = q^j$, $0 < q < a_0$, $\mathbb{D}_j = \mathbb{D}_{a_j}^{n-1}$, $\mathbb{A}_j = \mathbb{T}^{n-1} \times \mathbb{D}_j$, $\mathbb{D}_{j,\gamma} = \mathbb{D}_{a_j,\gamma}^{n-1}$, $\gamma_j^0 = \gamma_0(a_j) = C_5 a_j^{1+lb}$ and let $S_j(\theta, I)$ be the function $S(\theta, I)$

defined by Theorem 5.2 for $a = a_j$. Choose Γ' of the same type as Γ, $\Gamma \subset \Gamma' \subset \bigcup_{j=1}^{\infty} \mathbb{D}_j$, Γ being defined by (2.4) with some constants C_k^0, $k = 1, \ldots, 4$, while $\mathbb{D}_j$ are defined by (4.26) with some constants C_k, $k = 1, \ldots, 4$. Denote by $\mathbb{D}_j'$ the set (4.26) with constants C_k^0 and $a = a_j$. We can suppose that $\mathbb{D}_j \cap \mathbb{D}_{j+\nu} = \varnothing$ for any j and $\nu \geq 2$ and that $\mathbb{D}_j' \cap \mathbb{D}_{j+1}'$ has a nonempty interior.

Proposition 5.6. *There exist functions K_j, S_j defined by Theorem 5.2 in A_j with $\gamma_j^0 = C_5 a_j^{1+lb}$ and such that*

$$K_j(I) = K_{j+1}(I) \quad \text{in } \mathbb{D}_j \cap \mathbb{D}_{j+1}, \tag{5.27}$$

$$S_j(\theta, I) = S_{j+1}(\theta, I) \tag{5.28}$$

for any $I \in \mathbb{D}_{j,\gamma} \cap \mathbb{D}_{j+1}$ and $\gamma \geq \gamma_j^0$.

Proof. Denote by $\widetilde{T}_j$ the corresponding canonical transformation defined by Theorem 5.4. From Remark 1 after Pöschel's Theorem A [19] it follows that $\widetilde{T}_j^{-1} \circ \widetilde{T}_{j+1}(y, \eta) = (y + g(\eta), \eta)$ for $\eta \in \mathbb{D}^n_{a_j, \gamma_j^1} \cap \mathbb{D}^n_{a_j+1}$. Suppose that the exact symplectic map $T = \widetilde{T}_j^{-1} \circ \widetilde{T}_{j+1}$ is defined by the generating function $R(Y, \eta)$. Then $\operatorname{grad}_Y R(Y, \eta)$ does not depend on $Y \in \mathbb{T}^n$ for $\eta \in \mathbb{D}^n_{j,\gamma} \cap \mathbb{D}^n_{j+1}$, $\gamma \geq \gamma_j^1$ and since $R(Y, \eta)$ is small enough, we obtain $R(Y, \eta) = R(\eta)$. Thus $\widetilde{T}_{j+1}(y, \eta) = \widetilde{T}_j(y + \operatorname{grad} R(\eta), \eta)$, $\eta \in \mathbb{D}^n_{j,\gamma} \cap \mathbb{D}^n_{j+1}$.

Moreover, the function $\widetilde{S}_j(Y, \eta) + R(\eta)$ generates the map $\widetilde{T}_j \circ (\mathrm{Id} + \operatorname{grad} R, \mathrm{Id})$.

Let $\varphi_j = \varphi_j(y, \eta)$ solve the equation

$$\varphi - \frac{\partial \widetilde{S}_j}{\partial \eta}(\varphi, \eta) - \frac{\partial R}{\partial \eta}(\eta) = y$$

and let $\varphi_{j+1} = \varphi_{j+1}(y, \eta)$ solve $\varphi - \partial \widetilde{S}_{j+1}(\varphi, \eta)/\partial \eta = y$.

Then

$$\begin{aligned}(\varphi_{j+1}(y, \eta), \eta - (\partial \widetilde{S}_{j+1}/\partial Y)(\varphi_{j+1}(y, \eta), \eta)) \\ = \widetilde{T}_{j+1}(y, \eta) = (\varphi_j(y, \eta), \eta - (\partial \widetilde{S}_j/\partial Y)(\varphi_j(y, \eta), \eta))\end{aligned}$$

and we have

$$\varphi_{j+1} = \varphi_j, \qquad \operatorname{grad}_Y(\widetilde{S}_{j+1}(Y, \eta) - \widetilde{S}(Y, \eta)) = 0 \tag{5.29}$$

for $\eta \in \mathbb{D}^n_{j,\gamma} \cap \mathbb{D}^n_{j+1}$. As in the proof of Theorem 5.2 we have

$$K_{1,j}(Y, \eta) = \widetilde{H}(Y, \eta - (\partial \widetilde{S}_j/\partial Y)(Y, \eta))$$

and

$$K_{2,j}(\eta) = \widetilde{H}(0, \eta - (\partial \widetilde{S}_j/\partial Y)(0, \eta)).$$

In view of (5.29) we have $K_{2,j}(\eta) = K_{2,j+1}(\eta)$, $\eta \in \mathbb{D}^n_{j,\gamma} \cap \mathbb{D}^n_{j+1}$. Now instead of $K_{2,j+1}(\eta)$ we consider $\widetilde{K}_{2,j+1}(\eta) = K_{2,j+1}(\eta) + g(\eta)$ where $g(\eta) = K_{2,j}(\eta) - K_{2,j+1}(\eta)$ in $\mathbb{D}^n_j \cap \mathbb{D}^n_{j+1}$, $g(\eta) = 0$ on $\mathbb{D}^n_{j+1,\gamma} \cup (\mathbb{D}^n_{j+1} \cap \mathbb{D}^n_{j+2})$ and

$$\|g\|_{p,\mathbb{D}_{j+1,\gamma}} \le C_p a_{j+1}^{N-1/2-(l+1)b}.$$

According to Remark 5.1 we can replace $K_{2,j+1}$ by $K_{2,j+1} + g$. Thus we can suppose that $K_{2,j+1} = K_{2,j}$ on $\mathbb{D}^n_j \cap \mathbb{D}^n_{j+1}$. Then $\tau_{j+1}(I) = \tau_j(I)$ in $\mathbb{D}_{j,\gamma} \cap \mathbb{D}_{j+1}$ and so in $\mathbb{D}_j \cap \mathbb{D}_{j+1}$. Therefore,

$$S_{j+1}(\theta, I) = \widetilde{S}_{j+1}(\theta, 0, I, -\tau_{j+1}(I)) = \widetilde{S}_j(\theta, 0, I, -\tau_j(I)) = S_j(\theta, I)$$

for $I \in \mathbb{D}_{j,\gamma} \cap \mathbb{D}_{j+1}$, $\gamma \ge \gamma_j^1$ and any $\theta \in \mathbb{T}^{n-1}$. This completes the proof of Proposition 5.6.

Remark 5.2. Making use of the representation (5.5) and approximating the smooth functions Q_α, Q' with real-analytic ones, we can choose the sequence of analytic functions approximating the smooth Hamiltonian in the proof of Pöschel's Theorem B [19] independent of the domain. Thus equalities (5.27) and (5.28) may be assumed fulfilled by construction for any $I \in \mathbb{D}_j \cap \mathbb{D}_{j+1}$ and $\theta \in \mathbb{T}^{n-1}$.

Now we shall prove Theorem 1. We can find smooth functions $\varphi_j \in C_0^\infty(\mathbb{D}_j)$ such that $\sum_{j=1}^\infty \varphi_j(I) = 1$ for $I \in \Gamma'$ and $\varphi_j = 1$ on $\mathbb{D}'_j$,

$$|D^\beta \varphi_j| \le C_\beta a_j^{-|\beta|}. \tag{5.30}$$

Denote $S(\theta, I) = \sum_{j=1}^\infty \varphi_j(I) S_j(\theta, I)$ and by $K(I)$ the smooth function K such that $K(I) = K_j(I)$ in $\mathbb{D}_j$, $\tau(I) = -\frac{2}{3}(K(I))^{3/2}$. Then $S(\theta, I) = S_j(\theta, I)$ in $\mathbb{D}_{j,\gamma_j^0}$. For $\mu > 0$ and $\sigma > n-1$ denote by Γ_μ the set

$$\Gamma_\mu = \{I \in \Gamma; |\langle \operatorname{grad} \tau(I), k\rangle - k_n| \ge \mu I_1^{1+lb} |k|^{-\sigma} \quad \text{for any } k = (k', k_n) \in \mathbb{Z}^n \setminus \{0\}\},$$

$E = \overline{\Gamma}_\mu = \Gamma_\mu \cup \{(0, t_0)\}$. We choose C_5 in the definition of γ_j so that $\Gamma_\mu \cap \mathbb{D}_j \subset \mathbb{D}_{j,\gamma}$ for any $\gamma \le \gamma_{j-1}^0$. This is satisfied if $\mu(C_1 a_j)^{1+lb} \ge C_5 a_{j-1}^{1+lb}$, i.e. $C_5 \le \mu(C_1 q)^{1+lb}$. Then we have (5.30) and choosing $b < 1/(4l+8)$ in an appropriate way, we can prove the desired smoothness of K, g and U.

Now suppose that $N \ge 3$. From estimate (5.3) we obtain

$$\|K_j - \zeta_0\|_{2,\mathbb{D}_j} \le C_2 a^{N-5/2-(3l+7)b}$$

and the right-hand side can be estimated by $C_2 a^b$ if $b < 1/(6l+16)$. In particular,

$$\|\tau - H^0\|_{2,\mathbb{D}_j} \le \widetilde{C}_2 a^b. \tag{5.31}$$

Let $I \in \Gamma \setminus \Gamma_\mu$. Then $I \in \mathbb{D}_j$ for some j and $\tau_I(I) \in \Omega_{a_j} \setminus \Omega_{a_j}^{\gamma_j}$ where $\gamma_j = \mu(C_2 a_j)^{1+lb}$. Then by Lemma 5.1 we obtain

$$\operatorname{mes}(\Omega_{a_j} \setminus \Omega_{a_j}^{\gamma_j}) \le C\gamma_j a_j^{n-3+b} \le C' a_j^{n-2+(l+1)b}.$$

Then

$$\begin{aligned}\operatorname{mes}(\mathbb{D}_j \setminus \mathbb{D}_j^{\gamma_j}) &= \int_{\Omega_{a_j}\setminus\Omega_{a_j}^{\gamma_j}} |\det \tau_I^{-1}(\omega)|\, d\omega \\ &\le \operatorname{mes}(\Omega_{a_j} \setminus \Omega_{a_j}^{\gamma_j})(\min_{\mathbb{D}_j} |\det \tau_I(I)|)^{-1} \le C a_j^{n-2+(l+4-n)b}\end{aligned}$$

in view of (4.29) and (5.31). Here and below we denote by C, C' various constants. Thus we obtain

$$\begin{aligned}\operatorname{mes}(\Gamma \setminus \Gamma_\mu) &\le \sum_{j=1}^{\infty} \operatorname{mes}(\mathbb{D}_j \setminus \mathbb{D}_j^{\gamma_j}) \le C \sum_{j=1}^{\infty} a_j^{n-2+(l+4-n)b} \\ &= C\sum_{j=1}^{\infty} (q^j)^{n-2+(l+4-n)b} < C' q^{n-2+(l+4-n)b}.\end{aligned}$$

On the other hand,

$$\operatorname{mes}\Gamma \ge C \sum_{j=1}^{\infty} a_j^{n-2+2b} > C q^{n-2+2b},$$

hence $\operatorname{mes}(\Gamma\setminus\Gamma_\mu)/\operatorname{mes}\Gamma \le C' q^{(l+2-n)b} \le C' q^b$ provided that $l+2-n \ge 1$, i.e. $l \ge n-1$. This estimate immediately implies (2.6) and Theorem 1 is proved.

6. The two-dimensional case

Proof of Theorem 2. Let ζ be an approximate interpolating Hamiltonian for the billiard ball map B [14], i.e.

$$B(\rho) = \exp(-\zeta^{1/2} X_\zeta)(\rho) + O(\zeta^\infty)$$

in a neighbourhood of $S_+^*\partial\Omega$. Denote by M_r the closed curve $M_r = \{\zeta = r\}$ for $r \in (-\delta, \delta)$, δ small enough. For any $\rho \in M_r$ consider the map $\mathbb{R}^1 \ni t \to \exp(tX_\zeta)(\rho) \in M_r$ and denote by $l(r)$ its period. Let S be a transversal to M_0. For $\rho \in M_r$ denote by $t(\rho)$ the smallest positive time t such that $\exp(-tX_\zeta)(\rho) \in S$. Then $(t(\rho), \zeta(\rho))$, $0 \le t(\rho) \le l(\zeta(\rho))$ are symplectic coordinates in a neighbourhood of $S_+^*\partial\Omega$. We define $\varphi(\rho) = 2\pi t(\rho)/l(\zeta(\rho))$ and complement it to exact symplectic coordinates $(\varphi, I) \in T^*(\mathbb{T}^1)$. For this purpose we seek for a function $I = g(\zeta)$ so that the change $\chi: T^*\partial\Omega \to T^*(\mathbb{T}^1)$ transforms the symplectic 1-form $\sigma = \xi\, ds$ on Σ into $I d\varphi$. Then I and ζ satisfy the relation

$$(6.1) \qquad 2\pi I(\rho) = -\int_0^{\zeta(\rho)} l(r)\, dr + l_0,$$

$l_0 = \int_{\gamma^+} \sigma = l(0) = 2\pi\mathfrak{l}$, where γ^+ is $S^*_+\partial\Omega$ with the positive orientation. Denote by $\zeta(I)$ the solution of (6.1) with respect to ζ with initial condition $\zeta(\mathfrak{l}) = 0$ and by $\zeta_M(I)$ its Taylor expansion up to degree M which will be chosen large enough.

Now, in the coordinates (φ, I) the exact symplectic map $B_0 = \chi B \chi^{-1}$ can be written as

$$B_0(\varphi, I) = (\varphi + \tau'_M(I), I) + O((\mathfrak{l} - I)^{M+3/2}),$$

$\tau_M = -\frac{2}{3}\zeta_M^{3/2}(I)$ in a neighbourhood of $\mathbb{T}^1 \times \{\mathfrak{l}\}$. As above, we first consider B_0 in the sets

$$\mathbb{A}_j = \mathbb{T}^1 \times [\mathfrak{l} - C_1 a_j, \mathfrak{l} - C_2 a_j] = \mathbb{T}^1 \times \Gamma_j, \qquad j = 1, 2, \dots,$$

where $0 < C_2 < C_1$, $a_j = 4^{-j}$ and $\mathbb{A} = \bigcup_{j=1}^{\infty} \mathbb{A}_j$. We choose $M_j = N + 3j/2$, N large enough, and consider B_0 in $\mathbb{A}_j$ as the perturbation of the map $(\varphi, I) \to (\varphi + \tau'_{M_j}(I), I)$. By arguments similar to those in the proof of Theorem 5.2, replacing l, b and N by j, $1/2$ and $N+(3j+1)/2$ respectively, we construct exact symplectic transformations $U_j : \mathbb{A}_j \to \mathbb{A}_j$ such that $B_j = U_j^{-1} B U_j$ are generated by the functions $\tau_j(I) + H_j(\varphi, I)$, $j = 1, 2, \dots$, $H_j(\varphi, I) = 0$ for $(\varphi, I) \in \mathbb{T}^1 \times E_j$ where E_j are the preimages of the sets R_j (see (2.8)) under τ'_j. Here for brevity we write τ_j instead of τ_{M_j}. Now we choose $E = \bigcup_{j=1}^{\infty} E_j$. We have

$$E_j \subset \{I \in \Gamma_j ; |\tau'_j k_1 - k_2| \geq \mu' |I - \mathfrak{l}|^{1+j/2} |k|^{-\sigma}, \forall k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}\}$$

with some $\mu' > 0$, $\sigma > 1$ and repeating some details of the proof of (2.6), we obtain (2.7).

We can glue K_j, H_j and U_j together using again Remark 1 after Pöschel's Theorem A [16] which in the case $n = 2$ is quite obvious. Moreover,

$$|\partial_I^\beta (K - \zeta_{M_j})| \leq C_\beta |I - \mathfrak{l}|^{N-1-5\beta/2+j(1-\beta/2)}, \qquad I \in \Gamma_j,$$

thus the function K is C^∞ in Γ.

This completes the proof of Theorem 2.

Denote $R = \{\omega(I) = \tau'(I);\ I \in E\}$ and by $\omega \to I(\omega)$ the inverse map of the diffeomorphism $[\mathfrak{l} - \delta_0, \mathfrak{l}) \ni I \to \omega(I)$. Set $\Lambda_\omega = U(\mathbb{T}^1 \times \{I(\omega)\})$. Then $\Sigma_R = U(\mathbb{T}^1 \times E)$ is the union of the invariant curves Λ_ω of B with rotation numbers $\rho(\Lambda_\omega) = \omega \in R$. It obviously satisfies the inequality

$$\operatorname{mes} \Sigma^\delta - \operatorname{mes}(\Sigma_R \cap \Sigma^\delta) \leq C_N \delta^N$$

for any $N > 0$, $\delta > 0$ where $\Sigma^\delta = \partial\Omega \times [1 - \delta, 1]$.

Corollary 6.1. *There exists an approximate interpolating Hamiltonian ζ for the billiard ball map B in a neighbourhood of $S^*_+\partial\Omega$ such that*

$$B(\rho) = \exp(-\zeta^{1/2} X_\zeta)(\rho)$$

for any $\rho \in \Sigma_R$.

Proof of Theorem 3. Let $\mathfrak{g} \in \Gamma(m,n)$, $\mathfrak{g} = \{g_1, \dots, g_n\}$, i.e. $B^j(g_1) = g_{1+j}$, $j = 1, \dots, n-1$, $B^n(g_1) = g_1$ and $\mathfrak{g}$ has a winding number m. Denote by x_j the reflection points on $\partial\Omega$ corresponding to g_j and set $g_j = (s_j, \sigma_j)$ where s_j is the value of the natural parameter on $\partial\Omega$ at x_j and σ_j is the cosine of the angle between the oriented line segment linking x_j with x_{j+1} and the oriented tangent line at x_j. Let d be the maximal length of the arcs linking x_j with x_{j+1}, $j = 1, \dots, n$, and then we have $d \leq ml_0/n < \delta l_0$. Thus we obtain $0 \leq 1 - \sigma_j \leq C(\delta)$, $C(\delta) > 0$ as $\delta > 0$ which means that for $\delta > 0$ small enough and $m/n < \delta$ we have $g_j \in U(\mathbb{A})$, $1 \leq j \leq n$, if $\mathfrak{g} \in \Gamma(m,n)$. From Birkhoff's theorem it follows that $\Gamma(m,n) = \emptyset$ for $n \geq n_0$, n_0 large enough.

We shall use the following lemma.

Lemma 6.2. *Let $(m_k, n_k) \in \mathbb{N}^2$, $k = 1, 2, \dots$, and $2\pi m_k/n_k \to \omega$ as $k \to \infty$. Let $\mathfrak{g}^k - \{g_1^k, \dots, g_{n_k}^k\} \in \Gamma(m_k, n_k)$ and $U^{-1}(g_j^k) = (\varphi_j^k, I_k^k)$ where the map U is given by Theorem 2. Then the sequence $\{I_j^k,\ 1 \leq j \leq n_k,\ k = 1, 2, \dots\}$ is convergent and tends to $I(\omega)$.*

Proof. Denote by $Q_1(\varphi, I)$ the first component of the function $Q(\varphi, I)$ in Remark 2.2 and $p(\varphi, I) = \tau'(I) + Q_1(\varphi, I)$. Then $p_I > 0$ for $0 < \text{I} - I < \varepsilon$, $\varepsilon > 0$ small enough. Let $\mathfrak{g} \in \Gamma(m,n)$ and $U^{-1}(g_j) = (\varphi_j, I_j)$, $g_j \in \mathfrak{g}$, $j = 1, \dots, n$, and let $I(\omega_1) < I_1 < I(\omega_2)$, $I(\omega_i) \in E$, $i = 1, 2$. Then $I(\omega_1) < I_j < I(\omega_2)$ for $j = 1, \dots, n$ and $\omega_i \in R$, $\omega_1 < \omega_2$. Moreover, for the rotation numbers of Λ_ω and $\mathfrak{g}$ we have $\rho(\Lambda_{\omega_i}) = \tau'(I(\omega_i)) = \omega_i$, $\rho(\mathfrak{g}) = 2\pi m/n$ and since p is monotonely increasing with respect to I for φ fixed, we easily obtain $\omega_1 < 2\pi m/n < \omega_2$.

Note that the set R has no isolated points, thus if there exists a sequence $\mathfrak{g}^k \in \Gamma(m_k, n_k)$, $k = 1, 2, \dots$ such that $I_{j_k}^k \notin (I(\omega) - \varepsilon,\ I(\omega) + \varepsilon)$, then $I_{j_k}^k \notin (I(\omega_1),\ I(\omega_2))$ for any k, j and some $I(\omega_i) \in E$ appropriately chosen, $I(\omega_1) < I(\omega) < I(\omega_2)$. Hence $2\pi m_k/n_k \notin (\omega_1, \omega_2)$ which contradicts the condition of the lemma.

By Theorem 2 B_0 is an exact symplectic map with a generating function $\tau(I) + G(\varphi, I)$. It is easy to see that for $\alpha_0 = I d\varphi$ we have $B_0^*\alpha_0 - \alpha_0 = df$ where $f(\varphi, I) = I\tau'(I) - \tau(I) + h(\varphi, I)$ and $h(\varphi, I) = IG_I(\varphi, I) - G(\varphi, I)$ vanishes on E. Since the L-spectrum of B is a symplectic invariant which coincides with the length spectrum of Ω (see [6]), we have

$$L(\mathfrak{g}) = \sum_{j=1}^{n} f(B_0^j(\varphi_1, I_1)) = \sum_{j=1}^{n} f(\varphi_j, I_j)$$

where $L(\mathfrak{g})$ is the length of the periodic geodesic in Ω corresponding to $\mathfrak{g} = \{g_1, \dots, g_n\} \in \Gamma(m,n)$, $(\varphi_j, I_j) = U^{-1}(g_j)$, provided that $m/n < \delta$ and δ is small enough.

Let $m_k/n_k \to \omega/2\pi$ and $\mathfrak{g}^k \in \Gamma(m_k, n_k)$. From Lemma 6.2 we have $|I_j^k - I(\omega)| \leq \varepsilon$ for $k \geq k(\varepsilon) > 0$ where $\varepsilon = \varepsilon(\omega)$ is chosen so that $0 < \varepsilon <$

$|I(\omega)-\mathfrak{l}|/2$. Then $\tau(I) = -\frac{2}{3}(K(I))^{3/2}$ is a function of class C^∞ in the interval $|I - I(\omega)| < \varepsilon$ and

$$|L(\mathfrak{g}^k)/n_k - (\omega I(\omega) - \tau(I(\omega)))| \leq C(\omega) \max |I_j^k - I(\omega)|.$$

Hence the sequence $L(\mathfrak{g}^k)/n_k$, $k = 1, 2, \dots$, is convergent and for any choice of $\mathfrak{g}^k \in \Gamma(m_k, n_k)$ we have

$$\lim_{k\to\infty} L(\mathfrak{g}^k)/n_k = S(\omega) \tag{6.2}$$

where $S(\omega) = \omega I(\omega) - \tau(I(\omega))$ is just the Legendre transform of $\tau(I)$. Thus $L(m, n)$, $m/n \leq \delta$, recover uniquely the Legendre transform of $\tau(I)$ in R. Since E has no isolated points, one can recover from (6.2) the function $\tau(I)$ for any $I \in E$.

Now let Ω_1 and Ω_2 be two strictly convex domains in $\mathbb{R}^2$, $\Sigma^i = B^*\partial\Omega_i$; let B_i be the respective billiard ball maps, $i = 1, 2$, and let the assumptions of Theorem 3 be satisfied. Let E_i be defined by $E_i = \{I \in (\mathfrak{l}_i - \delta, \mathfrak{l}_i];\ \tau_i'(I) \in R\}$, $\mathfrak{l}_i = l_0^i/2\pi$, $i = 1, 2$. It is easily seen that $l_0^1 = l_0^2$ since $l_0^i = \lim_{n\to\infty} L(\mathfrak{g}^{n,i})$, $\mathfrak{g}^{n,i} \in \Gamma^i(1, n)$.

Now for all $\omega \in R$ we have

$$S_1(\omega) = S_2(\omega), \quad S_i(\omega) = \omega I_i(\omega) - \tau_i(I_i(\omega)), \qquad i = 1, 2.$$

Since R has no isolated points, we can differentiate with respect to ω which yields $I_1(\omega) = I_2(\omega)$. Hence $E_1 = E_2$ and $\tau_1(I) = \tau_2(I)$ for $I \in E_1$, thus $B_{0,1}(\varphi, I) = B_{0,2}(\varphi, I)$ for any $(\varphi, I) \in \mathbb{T}^1 \times E_1$. Conjugating with U_i, $i = 1, 2$, we obtain the assertion of Theorem 3.

7. Concluding remarks

7.1. A question may arise whether there exist strictly convex hypersurfaces with closed elliptic geodesics satisfying conditions (2.1) and (2.3). We can answer this question positively at least in the case $n = 3$, perturbing arbitrarily little the metric on an ellipsoid with three different axes in an arbitrarily small neighbourhood of any point of the shortest or the longest ellipse and applying a classical result of Nirenberg (see [7]) on the isometrical embeddings of compact oriented surfaces with Riemannian metric with strictly positive curvature as smooth convex surfaces in $\mathbb{R}^3$.

7.2. Theorem 1 and a theorem of Birkhoff-Lewis type (cf. [16]) show that the closure of the periodic points of B in a neighbourhood of $\tilde{\mathscr{O}}$ has a positive measure.

7.3. The results obtained hold also for any generalised billiard ball map arising as a boundary map for a pair of glancing hypersurfaces (see [15]). Indeed, we used in the proof only some properties of the approximated interpolating Hamiltonian but not the specific structure of Σ.

APPENDIX

As noted in §5, Theorem 5.4 can be derived from Pöschel's Theorem A [19]. The constants participating in the estimates throughout [19] depend only on n, σ, λ, ρ, R and α, but not on the domain or the parameter γ, i.e. their dependence on R, and, in Theorem A, on ρ, is not stated explicitly. In our case we have $R = Ca^{-b}$, $0 < \rho \leq a/2$, the parameters a and b participating in the definitions of the domain $\mathbb{D}_a^n$ and the parameter γ as well, thus we have to follow the dependence of the various constants in [19] on R and, in Theorem A, on ρ as well.

The Main Lemma in [19] holds for $\delta > 0$ small enough depending only on n, σ, λ, R and α, but not on the domain, γ or ρ. We put $\delta \leq C_1 R^{-3}$ and follow the dependence of the constants c_2, c_3, c_8, c_{11} in estimates (i), (ii), (iii), (v) on R.

In the proof of the Main Lemma a version of the implicit function theorem is used to solve for a local diffeomorphism φ and a small perturbation $\hat{\varphi}$ the equation

$$\varphi(\pi(\zeta)) = \varphi(\zeta) + \hat{\varphi}(\zeta) \tag{A.1}$$

for a map π close to the identity I:

Lemma A.1 [19, Lemma 4.2]. *Let φ, $\hat{\varphi}$ be real analytic on the ρ-neighbourhood $\Lambda + \rho$ of a domain Λ in $\mathbb{C}^n$, with $|D\varphi|$, $|D\varphi^{-1}| \leq S$ there. If*

$$|\hat{\varphi}|_{\Lambda+\rho} \leq \rho/c \tag{A.2}$$

with a constant $c = c(n, S)$, then there exists a unique real analytic map

$$\pi\colon \Lambda \to \Lambda + \rho, \qquad |\pi - I|_\Lambda \leq 2S|\hat{\varphi}|_{\Lambda+\rho}$$

such that (A.1) *holds. In addition, analytic dependence on parameters carries over from φ, $\hat{\varphi}$ to π. Also, $\pi - I$ is 2π-periodic in ζ, if φ and $\hat{\varphi}$ are.*

Following the proof of the above lemma, we see that we can choose the constant $c = c(n, S)$ in (A.2) in the form $c = c(n)(1 + S^3)$. Further on, for any $\delta \leq C_1 R^{-3}$, R large enough, the Main Lemma holds and the constants c_2, c_3, c_{11} can be chosen independent of R while c_8 depends linearly on R.

From the Main Lemma we can derive Theorem B for the normalized value of $\gamma = 1 \leq \rho$. We find that the constant c_{12} in [19, estimate (4.20)] depends linearly on R while the constant c_{13} in [19, (4.23)] can be chosen independent of R. This provides an additional factor R^{-1} in the left-hand side of the first estimate (3.24) in [19] for $\gamma = 1$ while the second estimate (3.24) and (3.25) in [19] for $\gamma = 1$ remain unchanged, with constant c_β independent of R.

Now we pass from the normalized value of $\gamma = 1 \leq \rho$ to the general case $0 < \gamma \leq \rho$. We replace the functions F and G in the formulation of Theorem

B in [19] by $\widetilde{F}^0 = \gamma^{-2}F^0 \circ \sigma_\gamma$, $\widetilde{G} = \gamma^{-2}G \circ \sigma_\gamma$ respectively and apply Theorem B for $\gamma = 1$ and the same R to the functions $\widetilde{F}^0$ and $\widetilde{G}$. This gives us functions $\widetilde{\Phi}$, $\widetilde{F}$, and $\widetilde{\Gamma}$ satisfying equalities (3.22), (3.23) in [19] for $\widetilde{F}^0$, $\widetilde{G}$ instead of F^0, G as well as estimates (3.24), (3.25) in [19] for $\gamma = 1$, the left-hand side of the first inequality of (3.24) being multiplied by R^{-1} and the constant c_β independent of R. Now, the functions $\Phi = \sigma_\gamma \circ \widetilde{\Phi} \circ \sigma_\gamma^{-1}$, $F = \gamma^2 \widetilde{F} \circ \sigma_\gamma^{-1}$ and $\Gamma = \gamma\widetilde{\Gamma} \circ \sigma_\gamma^{-1}$ satisfy (3.22), (3.23) in [19] with the functions F^0, G. Moreover, estimates (3.24) in [19] hold for any $\gamma \leq \rho$, the left-hand side of the first estimate (3.24) being multiplied by R^{-1}. On the other hand, we see that the exponent of γ in the right-hand side of (3.25) in [19] must be -1 and not -2.

Now let the assumptions of Theorem A be satisfied, the left-hand side of inequality (3.3) in [19] being multiplied by R and $\delta \leq C_1 R^{-3}$, C_1 independent of R and ρ, i.e. the assumptions of Theorem 5.4 hold. Then from Theorem B [19] we find, on account of the correction in estimate (3.25), that instead of (3.7) in [19] the generating function S satisfies the estimate

$$\|S\|_{\widetilde{\beta\lambda}, \widetilde{\beta}; \gamma} \leq C_\beta \gamma^{-1} R^{\beta+1} \|H - H^0\|_{\beta\lambda+\lambda+\sigma; \gamma}$$

C_β independent of R and ρ, which corresponds to estimate (ii) in Theorem 5.4. Moreover, Theorem A provides a diffeomorphism $T\colon \mathbb{T}^n \times \Omega \to \mathbb{T}^n \times I$ transforming the Hamiltonian vector field X_H on $\mathbb{T}^n \times I$ into the vector field $T^* X_H$ on $\mathbb{T}^n \times \Omega$ so that $T^* X_H|_{\mathbb{T}^n \times \Omega} = \langle \omega, \partial/\partial\theta \rangle$. Now, in order to obtain Theorem 5.4, we replace S, γ, H, H_0 and I by $\widetilde{S}$, γ_1, H', H_0' and $\mathbb{D}_a^n$, respectively, and the canonical map $\widetilde{T}$ generated by $\widetilde{S}$ satisfies for $(y, \eta) \in \mathbb{A}^n_{a, \gamma_1}$ the equality $\widetilde{T}(y, \eta) = T(y, (\partial K'/\partial\eta)(\eta))$ where $K'(\eta)$ is the nondegenerate Hamiltonian in (i) of Theorem 5.4 ($K(P)$ in [19, (3.5)]).

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